Maximum Entropy of Cycles of Even Period

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Abstract. A finite fully invariant set of a continuous map of the interval induces a permutation of that invariant set. If the permutation is a cycle, it is called its orbit type. It is known that Misiurewicz-Nitecki orbit types of period \( n \) congruent to 1 (mod 4) and their generalizations to orbit types of period \( n \) congruent to 3 (mod 4) have maximum entropy amongst all orbit types of odd period \( n \) and indeed amongst all \( n \)-permutations for \( n \) odd. We construct a family of orbit types of period \( n \) congruent to 0 (mod 4) which attain maximum entropy amongst \( n \)-cycles.

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1. Introduction

In recent years much attention has been given to the question of finding \(n\)-permutations and \(n\)-cycles which attain maximum entropy amongst all \(n\)-permutations and \(n\)-cycles, respectively. Originally this question was posed by Misiurewicz and Nitecki [10] who showed that the maximum entropy for \(n\)-permutations and for \(n\)-cycles is asymptotic to \(\log(2n/\pi)\). To obtain this result they defined a family of \(n\)-cycles for \(n \equiv 1 \pmod{4}\). Later, Geller and Tolosa [4] extended this definition to all \(n\) odd and proved that this family actually attains maximum entropy amongst all \(n\)-permutations (and hence all \(n\)-cycles). Geller and Weiss [5] then went on to show that this family is unique up to duality. For the case where \(n\) is even, King [8, 9] and Geller and Zhang [6] have shown, independently, that there are exactly two families of \(n\)-permutations (neither of which is a family of \(n\)-cycles), up to duality, which attain maximum entropy amongst \(n\)-permutations.

In this paper we construct a family of \(n\)-cycles which attain maximum entropy amongst all \(n\)-cycles of period \(n \equiv 0 \pmod{4}\).

2. Preliminaries

In this section we introduce some standard notation and well known results.

Let \(f\) be a continuous map of a compact interval \(I\) into itself. The orbit under \(f\) of a point \(x \in I\) is the sequence \(\text{Orb}(x) = \{x, f(x), f^2(x), \ldots\}\) (note that \(\text{Orb}(x)\) is an invariant set for \(f\)). If \(x = f^m(x)\) for some \(m \in \mathbb{N}\), then \(\text{Orb}(x)\) is finite and the orbit is periodic.

Recall that a permutation on \(n\) letters is a bijective map \(\theta : \{1, \ldots, n\} \to \{1, \ldots, n\}\). If \(\theta\) has the property that for \(1 \leq p \leq n\), \(\theta^p(1) = 1\) if and only if \(p = n\), then \(\theta\) is a cycle.

We define \(P_n\) to be the set of all permutations on \(n\) letters and \(C_n\) to be the set of all cycles on \(n\) letters. We also let \(P = \bigcup_{n \geq 1} P_n\) and \(C = \bigcup_{n \geq 1} C_n\).

**Definition 2.1.** Let \(\theta \in P_n\). Then
1. the dual of \(\theta\) is the permutation \(\tilde{\theta} \in P_n\) where \(\tilde{\theta}(i) = n + 1 - \theta(n + 1 - i)\), for \(i \in \{1, \ldots, n\}\),
2. the reverse of \(\theta\) is the permutation \(\bar{\theta} \in P_n\) where \(\bar{\theta}(i) = \theta(n + 1 - i)\), for \(i \in \{1, \ldots, n\}\),
3. the dual of the reverse of \(\theta\) is the permutation \(\theta^* \in P_n\) where \(\theta^*(i) = n + 1 - \theta(i)\), for \(i \in \{1, \ldots, n\}\),
4. the flip permutation \(\varphi \in P_n\) is defined as \(\varphi(i) = n + 1 - i\), for \(i \in \{1, \ldots, n\}\).

We can formulate \(\tilde{\theta}, \theta^*\) and \(\bar{\theta}\) in terms of the flip permutation \(\varphi\) as follows:

\[
\tilde{\theta}(i) = \varphi(\varphi(i)), \\
\theta^*(i) = \varphi(\theta(i)), \\
\bar{\theta}(i) = \varphi(\theta(\varphi(i))) = \varphi(\tilde{\theta}(i)) = \theta^*(\varphi(i)).
\]

If \(\theta \in C_n\), then \(\tilde{\theta} \in C_n\) but \(\bar{\theta}\) is not necessarily in \(C_n\).

If \(x\) has a periodic orbit under \(f\) of period \(n\) we can write \(\text{Orb}(x) = \{p_1, \ldots, p_n\}\) with \(p_1 < p_2 < \cdots < p_n\). This induces a cycle \(\theta \in C_n\) in the following way:

\[
\theta(i) = j \text{ if and only if } f(p_i) = p_j.
\]

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The cycle $\theta$ is called the orbit type of $\text{Orb}(x)$. More generally, if $S = \{p_1, \ldots, p_n\}$ (with $p_1 < p_2 < \cdots < p_n$) is any finite fully invariant set for $f$ (that is, $f(S) = S$), we can define the type of $S$ to be the permutation $\theta \in P_n$ with

$$\theta(i) = j \text{ if and only if } f(p_i) = p_j.$$ 

If $S$ is a fully invariant set for $f$ of type $\theta \in P_n$ then there is a unique map $f_\theta : [1, n] \to [1, n]$ satisfying

(i) $f_\theta(i) = \theta(i)$, for $i \in \{1, \ldots, n\}$,

(ii) $f_\theta$ is affine on each interval $I_i = \{x \in \mathbb{R} : i \leq x \leq i + 1\}$, for $i \in \{1, \ldots, n - 1\}$.

Note that $f_\theta$ is also a map with an invariant set of type $\theta$.

The map $f_\theta$ is called the linearisation of $f$ with respect to its invariant set of type $\theta$. Note that $f_\theta$ is entirely determined by $\theta$. It is a particularly simple map amongst those with an invariant set of type $\theta$. This also shows that every $\theta \in P_n$ is the type of an invariant set for some function $f$.

If, for each $i \in \{1, \ldots, n\}$, $f_\theta(i)$ is a local extremum of $f_\theta$ then $f_\theta$ is said to be maximodal. The permutation $\theta \in P_n$ is also said to be maximodal.

**Definition 2.2.** The entropy of a permutation $\theta \in P$ is

$$h(\theta) = \inf \{h(f)\}$$

where $f$ is a continuous map of a compact interval into itself which has an invariant set of type $\theta$ and $h(f)$ is the topological entropy of $f$ (see [2, pg 191]).

The determination of the entropy of $\theta$ is markedly simplified by the following result which can be found in Misiurewicz and Szlenk [11]:

**Proposition 2.3.** If $\theta \in P$ then $h(\theta) = h(f_\theta)$.

We note that for each $\theta \in P$, $h(\bar{\theta}) = h(\theta)$.

**Notation 2.4.** For $a, b \in \mathbb{R}$ with $1 \leq a \leq b \leq n - 1$,

$$[a, b] = \{m \in \mathbb{N} : a \leq m \leq b\};$$

$$O[a, b] = \{m \in \mathbb{N} : m \text{ is odd and } a \leq m \leq b\};$$

$$E[a, b] = \{m \in \mathbb{N} : m \text{ is even and } a \leq m \leq b\}.$$ 

**Notation 2.5.** For any $(n - 1) \times (n - 1)$ matrix $M$ with non-negative entries $m_{ij}$ we let

$$\left| M^{(j)} \right| = \sum_{i=1}^{n-1} m_{ij}$$

and

$$\|M\| = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} m_{ij} = \sum_{j=1}^{n-1} \left| M^{(j)} \right|.$$
Further simplification regarding the determination of $h(\theta)$ is obtained as follows:

**Definition 2.6.** The induced matrix $M(\theta)$ of $\theta \in P_n$ is the $(n - 1) \times (n - 1)$ matrix with $i j^{th}$ entry given by

$$a_{i,j} = \begin{cases} 1, & \text{if } f_\theta(I_i) \supset I_j \\ 0, & \text{otherwise,} \end{cases}$$

where $I_i = \{ x \in \mathbb{R} : i \leq x \leq i + 1 \}$ and $i, j \in [1, n - 1]$.

**Proposition 2.7.** If $\theta \in P$ then $h(\theta) = \log \rho(M(\theta))$, where $\rho(M(\theta))$ is the spectral radius of the induced matrix of $\theta$.

The proof of Proposition 2.7 can be found in Block and Coppel [2, Chapt VIII, Propn 19].

We wish to identify a cycle $\theta_n \in C_n$ which has maximum entropy. The following discussion helps to eliminate some members of $C_n$ which do not have maximum entropy.

If $\theta, \phi \in P$ we say that $\theta$ forces $\phi$ and we write $\theta \vdash \phi$ if every map which has an invariant set of type $\theta$ also has an invariant set of type $\phi$.

**Proposition 2.8.** The relation $\vdash$ is a partial preorder on $P$ and a partial order on $C$.

That is, $\vdash$ is reflexive and transitive on $P$ and in addition is anti-symmetric on $C$. The anti-symmetry property is due to Baldwin [1].

**Definition 2.9.** The permutation $\theta$ is forcing-maximal on a subset $S$ of $P$ if $\theta \in S$ and, for all $\phi \in S$, if $\phi \vdash \theta$ then $\theta \vdash \phi$.

If $\theta, \phi \in P$ and $\theta \vdash \phi$ then, from Definition 2.2, clearly $h(\theta) \geq h(\phi)$.

Since entropy respects the partial order in $C_n$, we need only consider those cycles which are forcing-maximal. The next result, due to Jungreis [7, Corollary 9.6], allows us to analyse this class more simply.

**Theorem 2.10.** If the cycle $\theta \in C_n$ is forcing-maximal then it is maximal.

3. Some useful properties of the induced matrix of a maximal permutation

The definition of the induced matrix $M(\theta)$ of a maximal permutation $\theta$ naturally gives rise to easily assimilated information regarding the rows of the matrix. In particular, the following result is easy to show:

**Remark 3.1.** Let $a_{i,j}$ be the entries in the induced matrix of a maximal permutation $\theta$, where $i, j \in [1, n - 1]$. If $f_\theta$ assumes a minimum at 1; that is, $\theta(1) < \theta(2)$, then

$$a_{i,j} = 1 \iff \begin{cases} \theta(i) \leq j \leq \theta(i + 1) - 1, & \text{if } i \text{ is odd} \\ \theta(i + 1) \leq j \leq \theta(i) - 1, & \text{if } i \text{ is even.} \end{cases}$$

If $f_\theta$ has a maximum at 1; that is, $\theta(1) > \theta(2)$, then

$$a_{i,j} = 1 \iff \begin{cases} \theta(i + 1) \leq j \leq \theta(i) - 1, & \text{if } i \text{ is odd} \\ \theta(i) \leq j \leq \theta(i + 1) - 1, & \text{if } i \text{ is even.} \end{cases}$$
We note that a trivial corollary of the above is that if $1 \leq i \leq n - 1$, $1 \leq j_1 < j_2 \leq n - 1$ and $a_{i_1} = a_{i_2} = 1$, then $a_{i} = 1$ for $j_1 \leq j \leq j_2$; that is, there are no "gaps" in rows of 1's. This idea in conjunction with Remark 3.1 leads to the even more useful corollary:

**Lemma 3.2 (The Shape Lemma).** Let $\theta$ be a maximodal permutation and denote the entries of the induced matrix $M(\theta)$ by $a_{ij}$. Suppose that for some indices $i, j$ we have

$$a_{i-1,j} = a_{i,j} = 1.$$ 

1. If $\theta(1) < \theta(2)$ then

$$a_{i-1,j'} = a_{i,j'},$$

for every $j' \leq j$ (respectively, $j' \geq j$) if $i$ is odd (respectively, $i$ is even).

2. If $\theta(1) > \theta(2)$ then

$$a_{i-1,j'} = a_{i,j'},$$

for every $j' \leq j$ (respectively, $j' \geq j$) if $i$ is even (respectively, $i$ is odd).

**Proof.** This is a direct consequence of Remark 3.1. To see how it works in the first of the four cases, assume that $\theta(1) < \theta(2)$, $i$ is odd and $a_{i-1,j} = a_{i,j} = 1$ for some $j$. We wish to show that $a_{i-1,j'} = a_{i,j'}$ for every $j' \leq j$.

By Remark 3.1, since $\theta(1) < \theta(2)$ and $i$ is odd we have

$$a_{i,j'} = 1 \iff \theta(i) \leq j' \leq \theta(i+1) - 1$$

and since $i - 1$ is even,

$$a_{i-1,j'} = 1 \iff \theta(i - 1 + 1) = \theta(i) \leq j' \leq \theta(i - 1) - 1.$$ 

Furthermore, since $a_{i-1,j} = a_{i,j} = 1$ we have

$$\theta(i) \leq j \leq \theta(i + 1) - 1 \quad \text{and} \quad \theta(i) \leq j \leq \theta(i - 1) - 1.$$ 

Thus for each $j' \leq j$, either

$$\theta(i) \leq j' \quad (\leq j) \quad \text{and so} \quad a_{i-1,j'} = a_{i,j'} = 1$$

or

$$1 \leq j' < \theta(i) \quad \text{and so} \quad a_{i-1,j'} = a_{i,j'} = 0.$$ 

The other three cases follow similarly. \hfill \Box

Remark 3.1 also permits us to retrieve $\theta$ from $M(\theta)$ by the following algorithm: If $\theta(1) < \theta(2)$ and $i$ is odd, then $j = \theta(i)$ is found by scanning the row $i$ from the left and stopping at the first 1, which corresponds to column $j$, whilst if $i$ is even then $j = \theta(i)$ is found by scanning row $i$ from the right and stopping one place before the first 1 (we note that this place will be $n$ if $a_{i,n-1} = 1$), which corresponds to column $j$. This also tells us that given columns $j$ and $j - 1$ of $M(\theta)$ for some $j \in [2, n - 1]$ we can find $i = \theta^{-1}(j)$. (Given column 1 we can find $\theta^{-1}(1)$ and given column $n - 1$ we can find $\theta^{-1}(n)$). An analogous algorithm holds if $\theta(1) > \theta(2)$.

Practically we also need information about the columns of $M(\theta)$. The Shape Lemma helps as it tells us about consecutive rows of $M(\theta)$. In fact a combination of the Shape Lemma and the fact that $\theta^{-1}(i)$ is unique, leads both to the well known result.
Corollary 3.3. Let $n$ be even and $\theta \in P_n$ be maximodal with induced matrix $B = M(\theta)$. The $j^{th}$ column sum $|B^{(j)}|$ of $B$ satisfies

$$|B^{(j)}| \leq \begin{cases} 2j, & \text{if } j < 2k \\ n - 1, & \text{if } j = 2k \\ 2(n - j), & \text{if } j > 2k \end{cases}$$

and the even stronger result

Corollary 3.4. Let $n$ be even and $\theta \in P_n$ be maximodal with induced matrix $B = M(\theta)$. If

$$|B^{(j')}| = 2j \text{ for some } j < 2k \text{ then } |B^{(j')}| = 2j' \text{ for all } j' \leq j$$

and if

$$|B^{(j')}| = 2(n - j) \text{ for some } j > 2k \text{ then } |B^{(j')}| = 2(n - j) \text{ for all } j' \geq j.$$

Corollary 3.3 may also be obtained from the following result due to Misiurewicz and Nitecki [10, Lemma 11.9]:

Proposition 3.5. If $\theta$ is a permutation of length $n$, then its induced matrix $M = M(\theta)$, of order $(n - 1) \times (n - 1)$, has $j^{th}$ column sum

$$|M^{(j)}| \leq \min \{2j, 2(n - j)\}.$$

4. The family of orbit types

In this section we define the orbit type $\theta_n$ of period $n = 4k$, $k > 1$, and state some general features of $\theta_n$ and its induced matrix $M(\theta_n)$. We will show that this family of orbit types achieves maximum entropy amongst all orbit types of period $n$. In the case $n = 4$, the cycle which achieves maximum entropy also achieves maximum entropy amongst 4-permutations and so is not considered here (see King [8]).

Definition 4.1. We define $\theta_n$ by

$$\theta_n: \quad j \mapsto \begin{cases} 2k - j + 1, & \text{if } j \in O[1, k + 1] \\ 2k - j + 2, & \text{if } j \in O[k + 2, 2k + 1] \\ j - 2k - 1, & \text{if } j \in O[2k + 3, 3k] \\ j - 2k, & \text{if } j \in O[3k + 1, n - 1] \\ 2k + j, & \text{if } j \in E[2, k + 1] \\ 2k + j - 1, & \text{if } j \in E[k + 2, 2k] \\ 6k - j + 2, & \text{if } j \in E[2k + 2, 3k] \\ 6k - j + 1, & \text{if } j \in E[3k + 1, n]. \end{cases}$$
An equivalent formulation for $\theta_n$, which is easier to use in some situations is

$$\theta_n : \begin{cases} 
2i - 1 \leftrightarrow 2k - 2i + 2, & \text{if } 1 \leq i \leq (k + 2)/2 \\
2k - 2i + 3 \leftrightarrow 2i - 1, & \text{if } 1 \leq i \leq (k + 1)/2 \\
2k + 2i + 1 \leftrightarrow 2i, & \text{if } 1 \leq i \leq (k - 1)/2 \\
n - 2i + 1 \leftrightarrow 2k - 2i + 1, & \text{if } 1 \leq i \leq k/2 \\
2i \leftrightarrow 2k + 2i, & \text{if } 1 \leq i \leq (k + 1)/2 \\
2k - 2i + 2 \leftrightarrow n - 2i + 1, & \text{if } 1 \leq i \leq k/2 \\
2k + 2i \leftrightarrow n - 2i + 2, & \text{if } 1 \leq i \leq k/2 \\
n - 2i + 2 \leftrightarrow 2k + 2i - 1, & \text{if } 1 \leq i \leq (k + 1)/2.
\end{cases}$$

For example, for $k$ odd, $\theta_n$ is easily seen to be the cycle

$$(2k + 1 \ 1 \ 2k \ n - 1 \ \ldots \ 2k - 2i + 3 \ 2i - 1 \ 2k - 2i + 2 \ n - 2i + 1 \ \ldots \ \ldots \ k + 4 \ k - 2 \ k + 3 \ 3k + 2 \ k + 2 \ k \ k + 1 \ 3k + 1 \ 3k \ k - 1 \ \ldots \ k - 3 \ 3k + 3 \ \ldots \ \underbrace{2k + 2i + 1}_{2 \leq i \leq \frac{k-3}{2}} \ 2i \ 2k + 2i \ n - 2i + 2 \ \ldots \ \ldots \ 2k + 3 \ 2 \ 2k + 2 \ n).$$

Similarly, for $k$ even, $\theta_n$ is the cycle

$$(2k + 1 \ 1 \ 2k \ n - 1 \ \ldots \ 2k - 2i + 3 \ 2i - 1 \ 2k - 2i + 2 \ n - 2i + 1 \ \ldots \ \ldots \ k + 3 \ k - 1 \ k + 2 \ 3k + 1 \ k + 1 \ k \ 3k \ 3k + 2 \ 3k - 1 \ k - 2 \ \ldots \ \underbrace{2k + 2i + 1}_{2 \leq i \leq \frac{k-2}{2}} \ 2i \ 2k + 2i \ n - 2i + 2 \ \ldots \ \ldots \ 2k + 3 \ 2 \ 2k + 2 \ n).$$

We also note the following general features of $f_{\theta_n}$:

1. The map $f_{\theta_n}$ has a local minimum at $j = 1$.
2. The map $f_{\theta_n}$ is maximodal and has all maximum values above all minimum values.
3. The map $f_{\theta_n}$ has a global minimum at $j = 2k + 1$.
4. The map $f_{\theta_n}$ has a global maximum at $j = 2k + 2$.

The general “shape” of the graph $f_{\theta_n}$ for $n \equiv 0 \pmod{4}$ is illustrated in Figures 1 and 2 below.
The permutation $\theta_n$ is not self dual, however the dual, $\overline{\theta_n}$, is automatically a cycle as it is conjugate to $\theta_n$ by $\varphi$. In general, the reverse permutation $\overline{\theta}$ of a cycle $\theta$ is not a cycle, but in the case of $\theta_n$, $\overline{\theta_n}$ is a cycle and hence so is $\theta^*$. Specifically, $\overline{\theta_n}$ is the cycle 

\[
\begin{align*}
(2k & \ 1 \ 2k + 1 \ n - 1 \ \ldots \ \frac{2k + 2i - 2}{2} \ 2i - 1 \ \frac{2k + 2i - 1}{2} \ n - 2i + 1 \ \ldots \ \\
& \ \ldots \ 3k - 3 \ k - 2 \ 3k - 2 \ 3k + 2 \ 3k - 1 \ k \ 3k \ 3k + 1 \ k + 1 \ k - 1 \\
& \ \frac{k + 2}{2} \ 3k + 3 \ \ldots \ \frac{2k - 2i}{2} \ 2i \ 2k - 2i + 1 \ n - 2i + 2 \ \ldots \\
& \ \ldots \ 2k - 2 \ 2 \ 2k - 1 \ n)
\end{align*}
\]

for $k$ odd, and for $k$ even, $\overline{\theta_n}$ is the cycle 

\[
\begin{align*}
(2k & \ 1 \ 2k + 1 \ n - 1 \ \ldots \ \frac{2k + 2i - 2}{2} \ 2i - 1 \ \frac{2k + 2i - 1}{2} \ n - 2i + 1 \ \ldots \ \\
& \ \ldots \ 3k - 2 \ k - 1 \ 3k - 1 \ 3k + 1 \ 3k \ k \ 3k + 2 \ k + 2 \ k - 2 \\
& \ \frac{k + 3}{2} \ 3k + 4 \ \ldots \ \frac{2k - 2i}{2} \ 2i \ 2k - 2i + 1 \ n - 2i + 2 \ \ldots \\
& \ \ldots \ 2k - 2 \ 2 \ 2k - 1 \ n).
\end{align*}
\]

We now present the main theorem of the paper.

**Theorem 4.2.** For $n \equiv 0 \pmod{4}$ the permutations $\theta_n$, $\overline{\theta_n}$ and their duals are cycles and have maximum entropy amongst all cycles of period $n$.

In the proof we will show that the cycle $\theta_n$ has maximum entropy and that $h(\theta_n) = h(\overline{\theta_n})$. It then follows that so do $\overline{\theta_n}$ and the dual of $\theta_n$ (see comments after Proposition 2.3).
We will now consider the induced matrix $B = M(\theta_n)$ of $\theta_n$. The rows of $B$ are
given by the following formulae:

**Proposition 4.3.** Let $n = 4k$ and $B = M(\theta_n)$. If $k$ is odd then

$$B_{j,r} = 1 \iff \begin{cases} 
2k - j + 1 \leq r \leq 2k + j, & \text{if } j \in O[1,k] \\
2k - j + 2 \leq r \leq 2k + j - 1, & \text{if } j \in O[k + 2,2k - 1] \\
1 \leq r \leq n - 1, & \text{if } j = 2k + 1 \\
j - 2k - 1 \leq r \leq 6k - j, & \text{if } j \in O[2k + 3,3k - 2] \\
k - 1 \leq r \leq 3k - 1, & \text{if } j = 3k \\
j - 2k \leq r \leq 6k - j - 1, & \text{if } j \in O[3k + 2,n - 1] \\
2k - j \leq r \leq 2k + j - 1, & \text{if } j \in E[2,k - 1] \\
k \leq r \leq 3k, & \text{if } j = k + 1 \\
2k - j + 1 \leq r \leq 2k + j - 2, & \text{if } j \in E[k + 3,2k] \\
j - 2k \leq r \leq 6k - j + 1, & \text{if } j \in E[2k + 2,3k - 1] \\
j - 2k + 1 \leq r \leq 6k - j, & \text{if } j \in E[3k + 1,n - 2]. 
\end{cases}$$

If $k$ is even then

$$B_{j,r} = 1 \iff \begin{cases} 
2k - j + 1 \leq r \leq 2k + j, & \text{if } j \in O[1,k - 1] \\
k \leq r \leq 3k, & \text{if } j = k + 1 \\
2k - j + 2 \leq r \leq 2k + j - 1, & \text{if } j \in O[k + 3,2k - 1] \\
1 \leq r \leq n - 1, & \text{if } j = 2k + 1 \\
j - 2k - 1 \leq r \leq 6k - j, & \text{if } j \in O[2k + 3,3k - 1] \\
j - 2k \leq r \leq 6k - j - 1, & \text{if } j \in O[3k + 1,n - 1] \\
2k - j \leq r \leq 2k + j - 1, & \text{if } j \in E[2,k] \\
2k - j + 1 \leq r \leq 2k + j - 2, & \text{if } j \in E[k + 2,2k] \\
j - 2k \leq r \leq 6k - j + 1, & \text{if } j \in E[2k + 2,3k - 2] \\
k + 1 \leq r \leq 3k + 1, & \text{if } j = 3k \\
j - 2k + 1 \leq r \leq 6k - j, & \text{if } j \in E[3k + 2,n - 2]. 
\end{cases}$$

We can now use these formulae to convert information about rows of $B$ to
information about columns of $B$.

**Definition 4.4.** Given $a, b \in \mathbb{N}$ with $1 \leq a \leq b \leq n - 1$, we define $\langle a, b \rangle$ to be
an $(n - 1) \times 1$ column matrix with $ith$ row entry $\langle a, b \rangle_i$, given by

$$\langle a, b \rangle_i = \begin{cases} 1, & \text{if } a \leq i \leq b \\
0, & \text{otherwise}. \end{cases}$$

If column $j$ of matrix $M$ is equal to $\langle a, b \rangle$ then we can write $M^{(j)} = \langle a, b \rangle$.

Note that $\|M^{(j)}\| = |M^{(j)}|$ (as in Notation 2.5) and if $M^{(j)} = \langle a, b \rangle$ then

$$\|M^{(j)}\| = b - a + 1.$$ 

Furthermore, for any $j \in [1,n - 1]$, we define $\langle M^{(j)} \rangle = \{i \in [1,n - 1] : m_{i,j} = 1\}$ and call this set the column support of $M^{(j)}$. 
Proposition 4.5. For \( n = 4k \) the column \( B^{(j)} \) is given by

\[
\langle 2k - j + 1, 2k + j \rangle, \quad \text{if } j \in O[1, 2k - 1]
\]
\[
\langle j - 2k, 6k - j - 1 \rangle, \quad \text{if } j \in O[2k + 1, 3k]
\]
\[
\langle j - 2k + 2, 6k - j + 1 \rangle, \quad \text{if } j \in O[3k + 1, n - 1]
\]
\[
\langle 2k - j + 2, 2k + j + 1 \rangle, \quad \text{if } j \in E[2, k - 1]
\]
\[
\langle 2k - j, 2k + j - 1 \rangle, \quad \text{if } j \in E[k, 2k - 2]
\]
\[
\langle 1, n - 1 \rangle, \quad \text{if } j = 2k
\]
\[
\langle j - 2k + 1, 6k - j \rangle, \quad \text{if } j \in E[2k + 2, n - 2].
\]

An equivalent formulation of Proposition 4.5 which is needed in our main proofs is

Proposition 4.6. For \( n = 4k \) the column \( B^{(j)} \) is given by

\[
\langle 2k - 2i + 2, 2k + 2i - 1 \rangle, \quad \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k/2
\]
\[
\langle 2k - 2i + 2, 2k + 2i + 1 \rangle, \quad \text{if } j = 2i \text{ and } 1 \leq i \leq (k - 1)/2
\]
\[
\langle 2i, n - 2i - 1 \rangle, \quad \text{if } j = 2k - 2i \text{ and } 1 \leq i \leq k/2
\]
\[
\langle 2i, n - 2i + 1 \rangle, \quad \text{if } j = 2k - 2i + 1 \text{ and } 1 \leq i \leq (k + 1)/2
\]
\[
\langle 1, n - 1 \rangle, \quad \text{if } j = 2k
\]
\[
\langle 2i - 1, n - 2i \rangle, \quad \text{if } j = 2k + 2i - 1 \text{ and } 1 \leq i \leq (k + 1)/2
\]
\[
\langle 2i + 1, n - 2i \rangle, \quad \text{if } j = 2k + 2i \text{ and } 1 \leq i \leq k/2
\]
\[
\langle 2k - 2i + 1, 2k + 2i \rangle, \quad \text{if } j = n - 2i \text{ and } 1 \leq i \leq (k - 1)/2
\]
\[
\langle 2k - 2i + 3, 2k + 2i \rangle, \quad \text{if } j = n - 2i + 1 \text{ and } 1 \leq i \leq k/2.
\]

From Proposition 4.5 we can easily obtain

Corollary 4.7. The matrix \( B \) has \( j \)th column sum

\[
|B^{(j)}| = \begin{cases} 
2j, & \text{if } j < 2k \\
n - 1, & \text{if } j = 2k \\
2(n - j), & \text{if } j > 2k.
\end{cases}
\]

Corollary 4.7 in conjunction with Corollary 3.3 shows that the matrix \( B \) has the maximum column sum for every column of \( B \). It is interesting to note that for \( n = 4k \) both the maximum entropy \( n \)-permutations (see King [8]) and maximum entropy \( n \)-cycles are maximodal, have all maximum values above all minimum values and

\[
\theta(2) < \theta(4) < \cdots < \theta(j) > \theta(j + 2) > \cdots > \theta(n) \quad \text{and}
\]
\[
\theta(1) > \theta(3) > \cdots > \theta(i) < \theta(i + 2) < \cdots < \theta(n - 1).
\]

where \( \theta(j) \) is the global maximum and \( \theta(i) \) is the global minimum. This is reflected in the induced matrices which have the property that each column contains the maximum number of 1's and there are no 0's between 1's in any column.

Recall that our task is to demonstrate that the cycles \( \theta_k \) and \( \theta \dot{\iota} \) and their duals all achieve maximum entropy amongst all cycles of period \( n \). As we know, for any
permutation $\theta$, $h(\theta)$ is given by $\log (\lim_{p \to \infty} \|M(\theta)^p\|^{1/p})$. It follows that if $\theta$ and $\phi$ are permutations, a sufficient but not necessary condition for $h(\theta)$ to be greater than or equal to $h(\phi)$ is that $\|M(\theta)^p\| \geq \|M(\phi)^p\|$ for all $p \in \mathbb{N}$. Given this, we will prove that $\|B^p\| \geq \|C^p\|$ for all $p \in \mathbb{N}$, where $B = M(\theta_n)$ and $C \in \Gamma$, for $\Gamma$ a class of matrices which we will define. Since the definition of $\Gamma$ is vital to the proof of our main theorem, we discuss it in some detail. To facilitate this discussion we introduce two simple definitions.

**Definition 4.8.** A matrix is called a 0-1 matrix if its only entries are 0 and 1.

**Definition 4.9.** If $X$ and $Y$ are any two $(n - 1) \times (n - 1)$, 0-1 matrices, $X$ is said to dominate $Y$ if and only if $\|X^p\| \geq \|Y^p\|$ for all $p \in \mathbb{N}$.

**Definition 4.10.** The permutation $\psi_n$ defined by:

$$
\psi_n : j \mapsto \begin{cases} 
2k - j + 1, & \text{if } j \in O[1, 2k - 1] \\
2k + j, & \text{if } j \in O[2k + 1, n - 1] \\
j - 2k, & \text{if } j \in O[2k + 1, n - 1] \\
6k - j + 1, & \text{if } j \in E[2k + 2, n].
\end{cases}
$$

is the unique $n$-permutation which has maximum entropy in $P_n$ and for which $f_{\psi_n}$ has a local minimum at 1 (see King [8, 9]).

**Notation 4.11.** The symbol $A$ is reserved to denote the induced matrix of $\psi_n$.

Recall, we have shown that we need only consider $n$-cycles which are maximal-modal. Furthermore, we will only consider $n$-cycles $\phi$ for which $f_\phi$ has a local minimum at 1 (the case where $f_\phi$ has a local maximum at 1 will be dealt with later). Our tactic is to define a class $\Gamma$ of $(n - 1) \times (n - 1)$, 0-1 matrices with the following properties:

1. For all $C \in \Gamma$, $B$ dominates $C$.
2. For all $n$-cycles $\phi$ for which $f_\phi$ is maximal-modal and has a local minimum at 1, there is a $C \in \Gamma$ such that $C$ dominates $M(\phi)$.

As dominance is clearly a transitive relationship, this will suffice to show that $h(\theta_n) \geq h(\phi)$ for any $n$-cycle $\phi$ for which $f_\phi$ is maximal-modal and has a local minimum at 1. In a later section we demonstrate that $\theta_n^\ast$, the dual of the reverse of $\theta_n$, is a cycle of identical topological entropy to $\theta_n$ and has the property that $h(\theta_n^\ast) \geq h(\phi)$ for any $n$-cycle $\phi$ for which $f_\phi$ is maximal-modal and has a local maximum at 1. As we only need to consider cycles which are maximal-modal, this will complete the proof of Theorem 4.2.

It turns out that $B \in \Gamma$ (a likely but not necessary consequence of properties 1 and 2) and in view of property 2, $\Gamma$ need not contain the induced matrices of all maximal-modal $n$-cycles for which $f_\phi$ has a local minimum at 1.

It is clear that we cannot permit $A$ to be an element of $\Gamma$ but the induced matrices of other $n$-permutations may be elements of $\Gamma$. In fact we only need to exclude the induced matrices of a relatively small class of these non-cyclic $n$-permutations from $\Gamma$. Those to be excluded arise from Proposition 4.12.
Proposition 4.12. Let $D$ be the induced matrix of an $n$-permutation $\phi$ such that $\phi$ is maximal with $\phi(1) < \phi(2)$ and

1. For some $i \in [1, (k-1)/2]$ the matrix $D$ is identical to the matrix $A$ on each of the four columns $2i$, $2k - 2i$, $2k + 2i$, $n - 2i$ or

2. For some $i \in [1, k/2]$ the matrix $D$ is identical to the matrix $A$ on each of the four columns $2i - 1$, $2k - 2i + 1$, $2k + 2i - 1$, $n - 2i + 1$.

Then $\phi$ is not a cycle.

Proof. Clearly it suffices to show that a proper subset of $[1, n]$ is fully invariant under $\phi$ to demonstrate that $\phi$ is not a cycle. If $D$ satisfies condition 1 for a specific $i \in [1, (k-1)/2]$ then the set $[2i + 1, 2k - 2i] \cup [2k + 2i + 1, n - 2i]$ is a proper subset of $[1, n]$ and is fully invariant under $\phi$. The former is trivial, since for $i \in [1, (k-1)/2]$, $[2i + 1, 2k - 2i] \cup [2k + 2i + 1, n - 2i] \subseteq [3, 2k - 2i] \cup [2k + 3, n - 2]$. The latter may easily be established in a case by case fashion using the method for retrieving $\phi$ from $D$ suggested in the comments following Lemma 3.2. To see how this works in a typical case, suppose for example that $j$ is an odd element of $[2i + 1, 2k - 2i] \cup [2k + 2i + 1, n - 2i]$. Trivially, $j \in O[2i + 1, 2k - 2i - 1] \cup O[2k + 2i + 1, n - 2i - 1]$. We wish to show that $\phi(j) \in [2i + 1, 2k - 2i] \cup [2k + 2i + 1, n - 2i]$. Recall that the method of retrieving $\phi(j)$ from $D$, where $j$ is odd, involves scanning row $j$ of $D$ from the left and stopping at the first 1. The number $\phi(j)$ is then the number of the column containing this 1. Note that since $j \in [2i + 1, 2k - 2i - 1] \cup [2k + 2i + 1, n - 2i - 1]$, $j$ is not in the column support $[2k - 2i, 2k - 2i - 1]$ of $D^{(2i)}$, but is in the column support $[2i, n - 2i - 1]$ of $D^{(2k - 2i)}$. In particular, this means $d_{2i} = 0$ and $d_{2k - 2i} = 0$ and hence as there are no "gaps" in rows of 1's in $D$, $2i < \phi(j) \leq 2k - 2i$. Thus $\phi(j) \in [2i + 1, 2k - 2i] \subseteq [2i + 1, 2k - 2i] \cup [2k + 2i + 1, n - 2i]$ as required.

The case where $j$ is even follows similarly with $D^{(2k-2i)} = \langle 2i + 1, n - 2i \rangle$ and $D^{(n-2i)} = \langle 2k - 2i + 1, 2k + 2i \rangle$ being used to show $\phi(j) \in [2k + 2i + 1, n - 2i]$. This completes the cases where $D$ satisfies condition 1 for a specific $i \in [1, (k-1)/2]$.

If the matrix $D$ satisfies condition 2 for a specific $i \in [1, k/2]$ then the set $S = [2i, 2k - 2i + 1] \cup [2k + 2i, n - 2i + 1]$ is a proper subset of $[1, n]$ and can be similarly shown to be fully invariant under $\phi$ by showing that if $j$ is an odd element of $S$ then $\phi(j) \in [2i, 2k - 2i + 1]$ and if $j$ is an even element of $S$ then $\phi(j) \in [2k + 2i, n - 2i + 1]$.

Proposition 4.12 identified non-cyclic $n$-permutations which have a fully invariant proper subset of a particular type. We will exclude the induced matrix of any such $n$-permutation from $\Gamma$. However, we will accept the induced matrix $C$ of any maximal $n$-permutation $\phi$ with $\phi(1) < \phi(2)$ and for which $C^{(2k)} = \langle 1, n - 1 \rangle$ as an element of $\Gamma$ provided $C$ does not satisfy the hypotheses of Proposition 4.12. That is, provided

(i) For all $i \in [1, (k-1)/2]$ at least one of the following inequalities holds:

- $C^{(2i)} \neq \langle 2k - 2i, 2k + 2i - 1 \rangle$ 
  \(= A^{(2i)} \neq B^{(2i)}\)

- $C^{(2k-2i)} \neq \langle 2i, n - 2i - 1 \rangle$ 
  \(= A^{(2k-2i)} = B^{(2k-2i)}\)

- $C^{(2k+2i)} \neq \langle 2i + 1, n - 2i \rangle$ 
  \(= A^{(2k+2i)} = B^{(2k+2i)}\)

- $C^{(n-2i)} \neq \langle 2k - 2i + 1, 2k + 2i \rangle$ 
  \(= A^{(n-2i)} = B^{(n-2i)}\).
(ii) For all $i \in [1, k/2]$ at least one of the following inequalities holds:

\[
C^{(2i-1)} \neq (2k - 2i + 2, 2k + 2i - 1) \quad \left( = A^{(2i-1)} = B^{(2i-1)} \right)
\]

\[
C^{(2k-2i+1)} \neq (2i, n - 2i + 1) \quad \left( = A^{(2k-2i+1)} = B^{(2k-2i+1)} \right)
\]

\[
C^{(2k+2i-1)} \neq (2i - 1, n - 2i) \quad \left( = A^{(2k+2i-1)} = B^{(2k+2i-1)} \right)
\]

\[
C^{(n-2i+1)} \neq (2k - 2i + 1, 2k + 2i - 2) \quad \left( = A^{(n-2i+1)} \neq B^{(n-2i+1)} \right).
\]

This makes it evident that $B$ belongs to $\Gamma$ but $A$ does not.

The maximodal $n$-cycles with $\phi(1) < \phi(2)$ of greatest concern to us are those whose induced matrices have columns which match $A$ on columns where $B^{(j)} \neq A^{(j)}$. To help us handle these cases we have the following result:

**Corollary 4.13.** If $D$ is the induced matrix of a maximodal $n$-cycle with $\phi(1) < \phi(2)$ such that

1. For some $i \in [1, (k - 1)/2]$, $D^{(2i)} = A^{(2i)} \neq B^{(2i)}$

or

2. For some $i \in [1, k/2]$, $D^{(n-2i+1)} = A^{(n-2i+1)} \neq B^{(n-2i+1)}$, then there exists $j' \in \{2k - 2i, 2k + 2i, n - 2i\}$ or $j' \in \{2k - 2i + 1, 2k + 2i - 1, 2i - 1\}$ in cases 1 and 2 respectively, such that $D^{(j')} \neq A^{(j')} = B^{(j')}$. 

The class $\Gamma$ of $(n - 1) \times (n - 1)$, 0–1 matrices we will consider is defined as follows:

**Definition 4.14.** The $(n - 1) \times (n - 1)$, 0–1 matrix $C \in \Gamma$ if and only if

1. $C^{(2k)} = (1, n - 1)$.

(So if $C$ is the induced matrix of a maximodal permutation $\phi$, all maximum values of $\phi$ are above all minimum values of $\phi$.)

2. If for some $i$ and $j_1 \leq j_2$ we have $c_{i,j_1} = c_{i,j_2} = 1$ then $c_{i,j} = 1$ for all $j \in [j_1, j_2]$.

(So $C$ has no row gaps.)

3. If $c_{i-1,j} = c_{i,j} = 1$ for some $j$ and some odd (respectively, even) $i$, then $c_{i-1,j'} = c_{i,j'}$ for all $j' \leq j$ (respectively, $j' \geq j$).

(So $C$ satisfies the Shape Lemma appropriate to induced matrices of maximodal permutations $\phi$ for which $\phi(1) < \phi(2)$. Note that in view of (1) we can conclude that for $i \in O[3, n - 1]$, $c_{i-1,j'} = c_{i,j'}$ for all $j' \leq 2k$ and for $i \in E[2, n - 2]$, $c_{i-1,j'} = c_{i,j'}$ for all $j' \geq 2k$.)

4. For all $j \neq 2k$

\[
|C^{(j)}| \leq |B^{(j)}| = |A^{(j)}| = \min \{2j, 2(n - j)\}.
\]

(So $C$ has the necessary permutation shape.)
5. 

(i) For all \( i \in [1, (k - 1)/2] \) at least one of the following inequalities holds:

\[
\begin{align*}
C^{(2i)} &\neq A^{(2i)} = (2k - 2i, 2k + 2i - 1), \\
C^{(2k-2i)} &\neq A^{(2k-2i)} = (2i, n - 2i - 1), \\
C^{(2k+2i)} &\neq A^{(2k+2i)} = (2i + 1, n - 2i), \\
C^{(n-2i)} &\neq A^{(n-2i)} = (2k - 2i + 1, 2k + 2i),
\end{align*}
\]

(ii) For all \( i \in [1, k/2] \) at least one of the following inequalities holds:

\[
\begin{align*}
C^{(2i-1)} &\neq A^{(2i-1)} = (2k - 2i + 2, 2k + 2i - 1), \\
C^{(2k-2i+1)} &\neq A^{(2k-2i+1)} = (2i, n - 2i + 1), \\
C^{(2k+2i-1)} &\neq A^{(2k+2i-1)} = (2i - 1, n - 2i), \\
C^{(n-2i+1)} &\neq A^{(n-2i+1)} = (2k - 2i + 1, 2k + 2i - 2).
\end{align*}
\]

(\textit{So \( C \) satisfies the cycle condition.})

Remark 4.15. Observe that \( B \) satisfies conditions 1–5 in the definition of \( \Gamma \) so \( B \in \Gamma \) whilst \( A \) satisfies conditions 1–4 but does not satisfy condition 5 so \( A \notin \Gamma \). In particular, the induced matrix \( M(\phi) \) of any maximodal \( n \)-permutation \( \phi \) with \( \phi(1) < \phi(2) \) and which has all maximum values above all minimum values satisfies conditions 1–4 of \( \Gamma \). So \( M(\phi) \in \Gamma \) if and only if it satisfies condition 5. A consequence of our proof that \( B \) dominates each element \( C \) of \( \Gamma \) will thus be that \( h(\theta_n) \geq h(\phi) \) if \( M(\phi) \) satisfies condition 5 (where \( \phi \) is maximodal, \( \phi(1) < \phi(2) \) and all maximum values of \( \phi \) are above all minimum values of \( \phi \)). Notice that \( \Gamma \) does not contain all possible induced matrices arising from maximodal \( n \)-cycles \( \phi \) with \( \phi(1) < \phi(2) \). This is because the induced matrix of such an \( n \)-cycle which also has at least one maximum value less than at least one minimum value will fail condition 1 of Definition 4.14. It will be seen in Lemma 5.4 in the next section that the maximodal \( n \)-cycles \( \phi \) for which \( \phi(1) < \phi(2) \) that are not elements of \( \Gamma \) are precisely those with at least one maximum value less than at least one minimum value. However this causes no problem since every such matrix is dominated by another element of \( \Gamma \).

5. Some easy lemmas

Throughout this section we will be dealing with \((n - 1) \times (n - 1)\), 0–1 matrices. Our aim is to present results which are helpful in the calculation of \( \|M^p\| \) or in comparing \( \|M^p\| \) with \( \|N^p\| \) for any \( p \in \mathbb{N} \). Lemma 5.5 and Corollaries 5.6, 5.7 and 5.13 are of particular importance.

Notation 5.1. Let \( M \) be an \((n - 1) \times (n - 1)\), 0–1 matrix.

1. For any \( T \subseteq [1, n - 1] \) and any \( p \in \mathbb{N} \)

\[
\sum_{j \in T} \left| (M^p)^{(j)} \right| \quad \text{will be denoted by} \quad M^p_T.
\]

Thus, for example, \( \|M^p\| = M^p_{[1, n-1]} \).

2. The set \( \{i \in [1, n - 1] : m_{ij} = 1\} \) will be denoted by \( S_j(M) \).
This latter notation will be of particular use in inductive arguments because of the next lemma.

**Lemma 5.2.** Let \( M \) be an \((n-1) \times (n-1)\), 0–1 matrix, then

\[
\left| (M^{p+1})^{(j)} \right| = \sum_{i \in S_j(M)} \left| (M^p)^{(i)} \right| = M^p_{S_j(M)}
\]

for all \( p \in \mathbb{N} \) and \( j \in [1, n-1] \).

Note that if we interpret \( M^0 \) as the \((n-1) \times (n-1)\) identity matrix, then Lemma 5.2 remains valid for \( p = 0 \).

**Proof.** Notice that if \( X \) is any \((n-1) \times (n-1)\) matrix and \( v \) is the 1 \times (n-1) matrix \((1, 1, 1, \ldots, 1)\) then \( vX \) is the 1 \times (n-1) matrix \((|X^{(1)}|, |X^{(2)}|, \ldots, |X^{(n-1)}|)\). Thus

\[
\left(\left| (M^{p+1})^{(1)} \right|, \left| (M^{p+1})^{(2)} \right|, \ldots, \left| (M^{p+1})^{(n-1)} \right| \right)
= vM^{p+1}
= (vM^p)M
= \left(\left| (M^p)^{(1)} \right|, \left| (M^p)^{(2)} \right|, \ldots, \left| (M^p)^{(n-1)} \right| \right) M.
\]

Thus

\[
\left| (M^{p+1})^{(j)} \right| = \sum_{i=1}^{n-1} \left| (M^p)^{(i)} \right| m_{i,j}
= \sum_{i \in S_j(M)} \left| (M^p)^{(i)} \right|,
\]

since

\[
m_{i,j} = \begin{cases} 1, & \text{if } i \in S_j(M) \\ 0, & \text{if } i \in [1, n-1] \setminus S_j(M). \end{cases}
\]

\[\]

**Corollary 5.3.** Let \( M \) and \( N \) be \((n-1) \times (n-1)\), 0–1 matrices. A sufficient condition for \( N \) to dominate \( M \) is that \( S_j(M) \subseteq S_j(N) \) for all \( j \in [1, n-1] \); (that is, that \( m_{i,j} = 1 \implies n_{i,j} = 1 \) for all \( i, j \in [1, n-1] \)).

**Proof.** It is easy to see by Lemma 5.2 and induction on \( p \) that \( \left| (M^p)^{(j)} \right| \leq \left| (N^p)^{(j)} \right| \) and hence that \( \|M^p\| \leq \|N^p\| \) for all \( p \in \mathbb{N} \). \[\]

We are now in a position to substantiate our claim at the end of the previous section.

**Lemma 5.4.** Let \( \phi \) be a maximodal \( n \)-cycle for which \( \phi(1) < \phi(2) \) and let \( M(\phi) \) be its induced matrix. Then there exists an element \( C \in \Gamma \) such that \( C \) dominates \( M(\phi) \).
PROOF. We know $M(\phi)$ has no row gaps and satisfies the Shape Lemma; that is, $M(\phi)$ satisfies conditions 2 and 3 in the definition of $\Gamma$. By Proposition 3.5, $M(\phi)$ satisfies condition 4 in the definition of $\Gamma$, and by Proposition 4.12, since $\phi$ is a cycle, $M(\phi)$ satisfies condition 5 in the definition of $\Gamma$. Therefore $M(\phi)$ is in $\Gamma$ if and only if $M(\phi)$ also satisfies condition 1. Clearly, if $M(\phi)^{(2k)} = (1, n - 1)$ then $M(\phi) \in \Gamma$ and setting $C = M(\phi)$ does the job. So assume $M(\phi)^{(2k)} \neq (1, n - 1)$.

By Remark 3.1 and as $\phi$ is maximal with $\phi(1) < \phi(2)$, $M(\phi)^{(2k)} = (1, n - 1)$ if and only if $\phi(i) \leq 2k$ for all $i \in O[1, n - 1]$ and $\phi(i) > 2k$ for all $i \in E[2, n]$. This means that all maximum values of $f_\phi$ are above all minimum values of $f_\phi$. Since we have assumed that $M(\phi)^{(2k)} \neq (1, n - 1)$ then there is at least one maximum value below one minimum value; that is, $\phi(i) \leq 2k$ for some $i \in E[2, n]$ and $\phi(i) > 2k$ for some $i \in O[1, n - 1]$.

So let $P = \phi^{-1}([2k + 1, n)) \cap O[1, n - 1] \neq \emptyset$ and $Q = \phi^{-1}([1, 2k]) \cap E[2, n] \neq \emptyset$ and note that $P$ and $Q$ have the same cardinality since $\phi$ is a bijection.

Given these preliminaries and making use of the information on $M(\phi)$ in Remark 3.1, we can now define the $(n - 1) \times (n - 1)$, $0 - 1$ matrix $C$ chosen to be a member of $\Gamma$ and to dominate $M(\phi)$ as follows:

(i) If $p \in P$; that is, $p \in O[1, n - 1]$ and $\phi(p) \geq 2k + 1$,

$$c_{p,j} = \begin{cases} 1, & \text{if } j \in [2k, \phi(p + 1) - 1] \\ 0, & \text{if } j \notin [2k, \phi(p + 1) - 1] \end{cases}$

while

$$M(\phi)_{p,j} = \begin{cases} 1, & \text{if } j \in [\phi(p), \phi(p + 1) - 1] \subset [2k, \phi(p + 1) - 1] \\ 0, & \text{if } j \notin [\phi(p), \phi(p + 1) - 1]. \end{cases}$$

If in addition, $p \neq 1$,

$$c_{p-1,j} = \begin{cases} 1, & \text{if } j \in [2k, \phi(p - 1) - 1] \\ 0, & \text{if } j \notin [2k, \phi(p - 1) - 1] \end{cases}$$

while

$$M(\phi)_{p-1,j} = \begin{cases} 1, & \text{if } j \in [\phi(p), \phi(p - 1) - 1] \subset [2k, \phi(p - 1) - 1] \\ 0, & \text{if } j \notin [\phi(p), \phi(p - 1) - 1]. \end{cases}$$

Note the implication here that if $p \in P$, $p + 1 \notin Q$ and further if $p \in P$ and $p \neq 1$, $p - 1 \notin Q$.

(ii) If $q \in Q$; that is, $q \in E[2, n]$ and $\phi(q) \leq 2k$,

$$c_{q-1,j} = \begin{cases} 1, & \text{if } j \in [\phi(q - 1), 2k] \\ 0, & \text{if } j \notin [\phi(q - 1), 2k] \end{cases}$$

while

$$M(\phi)_{q-1,j} = \begin{cases} 1, & \text{if } j \in [\phi(q - 1), \phi(q) - 1] \subset [\phi(q - 1), 2k] \\ 0, & \text{if } j \notin [\phi(q - 1), \phi(q) - 1]. \end{cases}$$

If in addition, $q \neq n$,

$$c_{q,j} = \begin{cases} 1, & \text{if } j \in [\phi(q + 1), 2k] \\ 0, & \text{if } j \notin [\phi(q + 1), 2k] \end{cases}$$
while
\[
M(\phi)_{qj} = \begin{cases} 1, & \text{if } j \in [\phi(q + 1), \phi(q) - 1] \subset [\phi(q + 1), 2k] \\ 0, & \text{if } j \notin [\phi(q + 1), \phi(q) - 1]. \end{cases}
\]

Note the implication here that if \( q \in Q, q - 1 \notin P \) and further if \( q \in Q \) and \( q \neq n, q + 1 \notin P \).

(iii) If \( i \in [1, n - 1] \) and \( c_{i,j} \) has not been specified in (i) or (ii) (that is, \( \{i, i + 1\} \cap (P \cup Q) = \emptyset \)) then
\[
c_{i,j} = M(\phi)_{i,j}.
\]

This may be more simply expressed without emphasizing the structure of \( M(\phi) \) by noting that the modification made to \( M(\phi) \) to produce \( C \) is to replace any \( 0 \) in the \( 2k \)th column by \( 1 \) and then to replace the minimum number of \( 0 \)'s remaining by \( 1 \)'s in order to ensure that there are no row gaps.

It is evident by this construction that \( C \) dominates \( M(\phi) \) and that \( C \) satisfies conditions 1, 2 and 3 in the definition of \( \Gamma \).

We now show that \( C \) satisfies condition 5. In view of the fact that \( M(\phi) \) satisfies condition 5, it suffices to show for each \( j \in [1, n - 1] \) that
\[
C^{(j)} = A^{(j)} \implies \left( M(\phi)^{(j)} = A^{(j)} \text{ or } C^{(n-j)} \neq A^{(n-j)} \right)
\]
or equivalently, that
\[
\left( C^{(j)} = A^{(j)} \text{ and } M(\phi)^{(j)} \neq A^{(j)} \right) \implies C^{(n-j)} \neq A^{(n-j)}.
\]

We will consider the case where \( j < 2k \). The case where \( j > 2k \) can be proved in similar fashion.

Thus let \( C^{(j)} = A^{(j)} \neq M(\phi)^{(j)} \) and \( j < 2k \). As \( C^{(j)} \neq M(\phi)^{(j)} \) and \( j < 2k \), the definition of \( C \), condition (ii) implies the existence of \( q \in Q \) such that \( \phi(q) - 1 < j \) with \( c_{q-1,j} = 1 \) and \( m_{q-1,j} = 0 \). Further, since \( n - j > 2k \), \( c_{q-1,n-j} = 0 \). In addition to this, if \( q \neq n \) then \( c_{q,j} = 1 \) and \( c_{n,n-j} = 0 \). As \( C^{(j)} = A^{(j)} = \langle x, y \rangle \) say, we have \( x \leq q - 1 \leq y \) and if \( q \neq n \), \( x \leq q \leq y \). Thus \( x + 1 \leq q \leq y + 1 \) and if \( q \neq n \) \( x - 1 \leq q - 1 \leq y - 1 \). Since
\[
A^{(n-j)} = \begin{cases} \langle x + 1, y + 1 \rangle, & \text{if } j \text{ is even} \\ \langle x - 1, y - 1 \rangle, & \text{if } j \text{ is odd} \end{cases}
\]
and if \( q \neq n \) both \( c_{q,n-j} = 0 \) and \( c_{q-1,n-j} = 0 \) and we see \( C^{(n-j)} \neq A^{(n-j)} \) as required.

This covers the case where \( q \neq n \). We show the case \( q = n \) does not arise as it leads to a contradiction.

If \( q = n \) we still have \( c_{n-1,j} = 1 \), \( c_{n-1,n-j} = 0 \) and \( x \leq n - 1 \leq y \). This latter inequality implies in turn that \( y = n - 1 \), \( x = 2 \) and \( j = 2k - 1 \). Thus
\[
A^{(n-j)} = A^{(2k+1)} = (1, n - 2)\).
\]
Note that since \( Q \neq \emptyset, P \neq \emptyset \); that is, there exists \( p \in O[1, n - 1] \) such that \( \phi(p) > 2k \), so as \( A^{(2k-1)} = A^{(2k-1)} = (2, n - 1) \), there is exactly one \( p \in O[1, n - 1] \) such that \( \phi(p) > 2k \) and indeed \( p = 1 \). Combining this with the fact that \( \phi(n - 1) < \phi(n) < 2k \) we conclude that there exists \( p' \in \)
$O[3, n - 3]$ such that $\phi(p') = 2k$. But then condition (iii) in the definition of $C$
 gives $c_{p'2k-1} = M(\phi)p'2k-1 = 0$ and $p' \in [2, n - 1]$. Hence the contradiction
and so $C^{(2k-1)} \neq A^{(2k-1)}$.

It remains to show that $C$ satisfies condition 4.

Remember we have assumed that $M(\phi)^{(2k)} \neq (1, n - 1)$ and hence $P$ and $Q$ are
sets of the same non-zero cardinality. Let the cardinality of $P$ and $Q$ be $m \geq 1$ and
let $P = \{p_1, p_2, \ldots, p_m\}$ and $Q = \{q_1, q_2, \ldots, q_m\}$ with the elements of $P$ and $Q$
listed in any order desired. For the specific listing chosen, define a new permutation
$\psi : [1, n] \to [1, n]$ as follows:

$\psi(j) := \phi(j)$, if $j \in [1, n] \setminus (P \cup Q)$
$\psi(p_i) := \phi(q_i)$, if $i = 1, 2, \ldots, m$
$\psi(q_i) := \phi(p_i)$, if $i = 1, 2, \ldots, m$.

Trivially, $\psi$ is a permutation with $f_{\psi}$ maximal, $\psi(1) < \psi(2)$ and all maximum
values of $f_{\psi}$ lie above all minimum values.

The now familiar analysis of $M(\psi)$ based on Remark 3.1 together with the
evident facts that for $p$ odd, $\psi(p) \leq \min \{2k, \phi(p)\}$ and for $q$ even, $\psi(q) \geq \max \{2k+1, \phi(q)\}$ show that $M(\psi)$ dominates $C$, which in turn dominates $M(\phi)$. Thus,
specifically for any $j$,

$$|C^{(j)}| \leq |M(\psi)^{(j)}| \leq \min \{2j, 2(n - j)\}$$

by Proposition 3.5.

Our basic aim now is to show that for any $C \in \Gamma$, $B = M(\theta_n)$ dominates $C$: that
is, $\|B^p\| \geq \|C^p\|$ for all $p \in \mathbb{N}$. We use induction to do this (Lemma 6.4).
Although we have some information about the structure of the matrices in $\Gamma$, we do not have
nearly enough to phrase our inductive hypothesis as simply “$\|B^p\| \geq \|C^p\|$” and so
our inductive hypothesis needs considerable strengthening. Clearly, the inductive hypothesis
“$\|B^{(j)}\| \geq \|C^{(j)}\|$” for all $j \in [1, n - 1]$ would be sufficient if it were
true. Unfortunately the result is not true for all $j \in [1, n - 1]$ but it is true for most
$j \in [1, n - 1]$ and this forms a central part of the inductive hypothesis (Lemma
6.4). For those $j \in [1, n - 1]$ where the condition fails we add other column sums
to $\|B^{(j)}\|$ in a variety of ways, so that the desired inequality is achieved.

The remainder of this section is devoted to building the machinery needed to
facilitate the establishment of the desired column sum comparisons. The next few
results are aimed at establishing Lemma 5.10.

**Lemma 5.5.** For any $0 - 1$ matrix $M$, if $S_j(M) \subseteq S_{j'}(M)$ then for all $p \in \mathbb{N} \cup \{0\}$,

$$\left|(M^p)^{(j)}\right| \leq \left|(M^p)^{(j')}\right|.$$

**Corollary 5.6.** If $C \in \Gamma$ and $1 \leq j \leq j' \leq 2k$ or $2k \leq j' \leq j \leq n - 1$ then
for all $p \in \mathbb{N} \cup \{0\}$,

$$\left|(C^p)^{(j)}\right| \leq \left|(C^p)^{(j')}\right|.$$
Corollary 5.7. If \( C \in \Gamma \), \( S \subseteq [1,n-1] \), \( T \subseteq [1,n-1] \) and there is an injection \( \rho : S \mapsto T \) with the property that for all \( j \in S \)
\[ j \leq \rho(j) \leq 2k \text{ or } 2k \leq \rho(j) \leq j, \]
then for all \( p \in \mathbb{N} \cup \{0\} \)
\[ C_S^p \leq C_T^p. \]

Notation 5.8. For \( j \in [1,n-1] \) and \( B = M(\theta_n) \), let
\[ X_j = \{ i \in [1,n-1] : b_{ij} = 1 \} = S_j(B) \]
and for any specific \( C \in \Gamma \) let
\[ Y_j = \{ i \in [1,n-1] : c_{ij} = 1 \} = S_j(C). \]
(Thus if the column \( B^{(j)} \) is equal to \( \langle x,y \rangle \) then \( X_j = \langle x,y \rangle \) and if the column \( C^{(j)} \) is equal to \( \langle a,b \rangle \) then \( Y_j = \langle a,b \rangle \). Of course in the case of \( C \), \( Y_j \) may not be a consecutive string of natural numbers.)

Observe here that by Lemma 5.2
\[ \left| (B^{p+1})^{(j)} \right| = B_{X_j}^p = \sum_{i \in X_j} \left| (B^p)^{(i)} \right| \]
and
\[ \left| (C^{p+1})^{(j)} \right| = C_{Y_j}^p = \sum_{i \in Y_j} \left| (C^p)^{(i)} \right|. \]

It follows that when we try to compare terms of the form \( B_{X_j}^p \) and \( C_{Y_j}^p \) the index sets \( X_j \) and \( Y_j \) play a central role. Clearly, if \( X_j = Y_j \) then our task is much simpler than if \( X_j \neq Y_j \) but this is unlikely to be the case since \( C \) is an arbitrary element of \( \Gamma \). However, if we can find a subset \( W \) of \([1,n-1]\) which is in a sense "closer" to \( X_j \) than \( Y_j \) is to \( X_j \) and which satisfies \( C_W^p \geq C_{Y_j}^p \), we may find the desired comparison easier to establish. The search for an appropriate \( W \) proceeds through Lemma 5.10, Corollary 5.11 and Lemma 5.12 to the general result in Corollary 5.13. Corollary 5.7 acts as the key tool in the proof of these results. To proceed on this path we first need to establish certain restrictions on \( Y_j \) in Lemma 5.9. The validity of these restrictions are trivial consequences of the Shape Lemma properties of elements of \( \Gamma \), (namely if \( C \in \Gamma \) then for \( i \in O[3,n-1] \), \( c_{i-1,j'} = c_{i,j'} \) for all \( j' \leq 2k \) and for \( i \in E[2,n-2] \), \( c_{i-1,j'} = c_{i,j'} \) for all \( j' \geq 2k \)).

Lemma 5.9. If \( C \in \Gamma \) and \( j \in [1,n-1] \) then
1. For \( j < 2k \),
   (a) \( Y_j \cap [2k,n-1] \) has an even number of elements, and
   (b) \( Y_j \cap [1,2k-1] \) has an even number of elements if and only if \( 1 \notin Y_j \).
2. For \( j > 2k \),
   (a) \( Y_j \cap [1,2k] \) has an even number of elements, and
   (b) \( Y_j \cap [2k+1,n-1] \) has an even number of elements if and only if \( n-1 \notin Y_j \).

The structure of an arbitrary column of a matrix \( C \in \Gamma \) can vary widely (except of course \( C^{(2k)} \)). This creates a difficulty when we try to compare \( \left| (C^p)^{(j)} \right| \) with \( \left| (B^p)^{(j)} \right| \). However we can make this task much simpler by defining sets (depending on \( j \)) which give us an upper bound on the size of \( \left| (C^p)^{(j)} \right| \). We need then only compare \( \left| (B^p)^{(j)} \right| \) to the appropriate upper bounds. The remainder of the results in this section establish these sets.
Lemma 5.10. If $C \in \Gamma$ and $j \in [1, n - 1]$, then

(i) For $j < 2k$ let $2\beta$ be the number of elements in $Y_j \cap [2k, n - 1]$ and let $V_j = [2k + 2\beta - 2j, 2k + 2\beta - 1] \cap [1, n - 1]$, and

(ii) For $j > 2k$ let $2\beta$ be the number of elements in $Y_j \cap [1, 2k]$ and let $V_j = [2k - 2\beta + 1, 2k - 2\beta + 2(n - j)] \cap [1, n - 1]$.

Then

$$0 \leq \beta \leq \min \{k, j, n - j\}$$

and for all $p \in \mathbb{N} \cup \{0\}$,

$$C^p_{Y_j} \leq C^p_{V_j}.$$

Proof. Since $2\beta$ is bounded above by the number of elements in $Y_j$, we have $2\beta \leq \min \{2j, 2(n - j)\}$ by condition 4 of $\Gamma$. Also $2\beta$ is bounded by either the number of elements in $[2k, n - 1]$ or the number of elements in $[1, 2k]$, which in either case is $2k$, thus $0 \leq \beta \leq \min \{k, j, n - j\}$. We now give the proof for $j < 2k$. The proof for $j > 2k$ is similar.

Let $j < 2k$. Note that the number of elements in $Y_j \leq 2j$ and that $Y_j \cap [2k, n - 1]$ has $2\beta$ elements. It follows that the number of elements in $Y_j \cap [1, 2k - 1]$ is bounded above by $\max \{2j - 2\beta, 2k - 1\}$. Note that $[2k, 2k + 2\beta - 1]$ has $2\beta$ elements and there is an injection $\rho_1 : Y_j \cap [2k, n - 1] \to [2k, 2k + 2\beta - 1]$ such that $2k \leq \rho_1(j) \leq j$.

Also, $[2k + 2\beta - 2j, 2k - 1] \cap [1, 2k - 1]$ has $\max \{2j - 2\beta, 2k - 1\}$ elements and there is an injection $\rho_2 : Y_j \cap [1, 2k - 1] \to [2k + 2\beta - 2j, 2k - 1] \cap [1, 2k - 1]$ such that $j \leq \rho_2(j) \leq 2k - 1$.

Combining these, there is an injection $\rho : Y_j \to V_j$ where $Y_j = (Y_j \cap [1, 2k - 1]) \cup (Y_j \cap [2k, n - 1])$ and $V_j = ([2k + 2\beta - 2j, 2k - 1] \cap [1, 2k - 1]) \cup [2k, 2k + 2\beta - 1]$, such that for all $j \in Y_j$, $j \leq \rho(j) \leq 2k$ or $2k \leq \rho(j) \leq j$.

Thus by Corollary 5.7, $C^p_{Y_j} \leq C^p_{V_j}$ for all $p \in \mathbb{N} \cup \{0\}$. 

Obviously $V_j$ is constructed from $Y_j$ by compressing elements of $Y_j$ towards the centre point $2k$ and by adding an appropriate number of elements so that $V_j$ still satisfies the conditions of Lemma 5.9, has no more elements than $X_j$ and the number of elements in $Y_j \cap [2k, n - 1]$ and $V_j \cap [2k, n - 1]$ (respectively $Y_j \cap [1, 2k]$ and $V_j \cap [1, 2k]$) are the same for $j < 2k$ (respectively $j > 2k$).

One point to note, which we will expand upon shortly, is that for each $j \neq 2k$ there is an appropriate value for $\beta$ such that $V_j = S_j(A)$. We wish to emphasize this fact by rephrasing and slightly strengthening Lemma 5.10 into the next corollary.
Corollary 5.11. Let \( C \in \Gamma \) and \( j \in [1, 2k - 1] \cup [2k + 1, n - 1] \) and let \( T_j \) be a collection of sets of consecutive integers defined as follows:

\[
T_j = \left\{
\begin{array}{l}
\{[2k - j + 1 + 2\alpha, 2k + j + 2\alpha] : \frac{(j-1)}{2} \leq \alpha \leq \frac{(j+1)}{2}\}, \text{ if } j \in O[1, k - 1] \\
\{[1, 2j - 1]\} \cup \left\{[2k - j + 1 + 2\alpha, 2k + j + 2\alpha] : \frac{(j+1-2k)}{2} \leq \alpha \leq \frac{(2k-j-1)}{2}\}, \text{ if } j \in O[k, 2k - 1] \\
\{[2k - j + 2\alpha, 2k + j - 1 + 2\alpha] : \frac{(j+1)}{2} \leq \alpha \leq \frac{(2k-j)}{2}\}, \text{ if } j \in E[2k, k - 1] \\
\{[1, 2j - 1]\} \cup \left\{[2k - j + 2\alpha, 2k + j - 1 + 2\alpha] : \frac{(j+1-2k)}{2} \leq \alpha \leq \frac{(2k-j)}{2}\}, \text{ if } j \in E[k, 2k - 2] \\
\{[2j - n + 1, n - 1]\} \cup \left\{[j - 2k + 2\alpha, 6k - j - 1 + 2\alpha] : \frac{(1+2k-j)}{2} \leq \alpha \leq \frac{(j-2k-1)}{2}\}, \text{ if } j \in O[2k + 1, 3k] \\
\{[j - 2k + 2\alpha, 6k - j - 1 + 2\alpha] : \frac{(j+1-n)}{2} \leq \alpha \leq \frac{(n-j+1)}{2}\}, \text{ if } j \in O[3k + 1, n - 1] \\
\{[2j - n + 1, n - 1]\} \cup \left\{[j - 2k + 1 + 2\alpha, 6k - j + 2\alpha] : \frac{(2k-j)}{2} \leq \alpha \leq \frac{(j-2k-2)}{2}\}, \text{ if } j \in E[2k + 2, 3k] \\
\{[j - 2k + 1 + 2\alpha, 6k - j + 2\alpha] : \frac{(j-n)}{2} \leq \alpha \leq \frac{(n-j)}{2}\}, \text{ if } j \in E[3k + 1, n - 2].
\end{array}\right.
\]

Then there exists \( V \in T_j \) such that for all \( p \in \mathbb{N} \cup \{0\} \)

\[ C_{\gamma_j}^p \leq C_{V}^p. \]

Proof. We will deal with the case where \( j \in O[1, 2k - 1] \). The other cases follow similarly. For \( j \in O[1, k - 1] \) we note by Lemma 5.10 that \( V_j \subseteq [2k + 2\beta - 2j, 2k + 2\beta - 1] \) for some \( \beta \) such that \( 0 \leq \beta \leq \min \{k, j, n-j\} = j \). If we set \( \alpha = (2\beta - 1 - j)/2 \), we note that \( \alpha \) is an integer, \( [2k - j +1 + 2\alpha, 2k + j + 2\alpha] = [2k + 2\beta - 2j, 2k + 2\beta - 1] \) and \( 0 \leq \beta \leq j \implies (-1 - j)/2 \leq \alpha \leq (j - 1)/2 \). Hence setting \( V = [2k + 2\beta - 2j, 2k + 2\beta - 1] \) gives

\[ C_{\gamma_j}^p \leq C_{V_j}^p \text{ (by Lemma 5.10)} \leq C_{V}^p \text{ (since } V_j \subseteq V). \]

(In fact \( V_j = [2k+2\beta-2j, 2k+2\beta-1] \) in the above case so actually \( V = V_j \).) On the other hand if \( j \in O[k, 2k-1] \) we have \( V_j = [2k+2\beta-2j, 2k+2\beta-1] \cap [1, n-1] \) for some \( \beta \) such that \( 0 \leq \beta \leq \min \{k, j, n-j\} = k \). Now if \( 0 \leq \beta \leq j-k \) we see \( V_j \subseteq [1, 2j-1] \) and we set \( V = [1, 2j-1] \). If \( j-k+1 \leq \beta \leq k \) we see \( V_j = [2k + 2\beta - 2j, 2k + 2\beta - 1] \) and if again we set \( \alpha = (2\beta - 1 - j)/2 \), \( \alpha \) is an integer, \( [2k - j +1 + 2\alpha, 2k + j + 2\alpha] = V_j \) and \( j-k+1 \leq \beta \leq k \implies (j - 1 - 2k)/2 \leq \alpha \leq (2k - 1 - j)/2 \) so we set \( V = V_j \).

Again since \( V_j \subseteq V \) we have \( C_{\gamma_j}^p \leq C_{V_j}^p \leq C_{V}^p \). \( \square \)

For the record, if \( j \in E[2, 2k-2] \) we set \( V = V_j \) if \( \beta \geq j-k+1 \) and \( V = [1, 2j-1] \) if \( \beta \leq j-k \) and set \( \alpha = (2\beta - j)/2 \). For \( j \in [2k + 1, n - 1] \) we set \( V = V_j \) if
\[ \beta \geq 3k - j + 1 \text{ and } V = [2j - n + 1, n - 1] \text{ if } \beta \leq 3k - j \text{ while we set} \]

\[
\alpha = \begin{cases} 
\frac{n-2\beta+1-j}{2}, & \text{if } j \text{ is odd} \\
\frac{n-2\beta-j}{2}, & \text{if } j \text{ is even.}
\end{cases}
\]

It is useful to note here that \( V \) is always constructed from \( V_j \) in such a way that \( V_j \subseteq V \). In fact, \( V_j = V \) unless

(i) \( j \in [k, 2k - 1], V = [1, 2j - 1] \) and \( V_j \) has less than \( 2j - 1 \) elements

or

(ii) \( j \in [2k + 1, 3k], V = [2j - n + 1, n - 1] \) and \( V_j \) has less than \( 2(n - j) - 1 \) elements.

We note that when \( \alpha = 0 \), the element \( V \) of \( T_j \) which arises is \( S_j(A) \) which is also the \( V_j \) constructed from \( Y_j \) in Lemma 5.10, and in general, \( V = S_j(A) \) if and only if \( V_j = S_j(A) \). This may create a problem. Recall that we eventually hope to use an inductive hypothesis to establish results like \( B_{X_j}^p \geq C_{V_j}^p \) for \( C \in \Gamma \). As \( \Gamma \) is still a large class of matrices, for any \( j \) there will be a large number of possibilities for \( Y_j \). The number of possibilities for \( V_j \) is considerably smaller, whilst the number of elements in \( T_j \) is either the same or smaller. In light of this we will generally choose to show that either \( B_{X_j}^p \geq C_{V_j}^p \) or \( B_{X_j}^p \geq C_{V_j}^p \) rather than \( B_{X_j}^p \geq C_{V_j}^p \), even though in some cases this is a much tougher problem. If \( V = V_j = S_j(A) \) showing that \( B_{X_j}^p \geq C_{V_j}^p \) is potentially the most difficult problem. (In most cases \( X_j = S_j(A) \), however the really difficult cases are when \( j \in E[2k - 1] \cup O[3k + 1, n - 1] \). Of course if \( Y_j = V_j = S_j(A) \) there is no way out, but if \( Y_j \neq V_j = S_j(A) = V \) we may be making trouble for ourselves by “replacing” \( Y_j \) by \( V \). We would still like to reduce the number of cases however, and the next lemma suggests an alternative “replacement” for \( Y_j \) to achieve this end.

**Lemma 5.12.** Let \( C, j, \Gamma \) and \( V_j \) be as in Lemma 5.10, and let \( U_j = \{ i \in [1, n - 1] : a_{i,j} = 1 \} = S_j(A) \) (so if column \( A^{(j)} \) is \( (u, v) \) then \( U_j = [u, v] \)). If \( j \neq 2k \) and \( Y_j \neq U_j \) but \( V_j = \bar{U}_j \), then for all \( p \in \mathbb{N} \cup \{ 0 \} \)

\[ C_{Y_j}^p \leq C_{V_j}^p \text{ or } C_{Y_j}^p \leq C_{V_j}^{p'} \]

where

(a) If \( j \in E[2, 2k - 2] \); that is, \( Y_j \neq U_j = V_j = [2k - j, 2k + j - 1] \), then

\[ V_j' = ([2k - j - 2, 2k - j - 1] \cup [2k - j + 2, 2k + j - 1]) \cap [1, n - 1] \]

(the intersection with \([1, n - 1] \) is only significant for \( j = 2k - 2 \)).

\[ V_j'' = [2k - j, 2k + j - 3] \cup [2k + j, 2k + j + 1] \]

(b) If \( j \in O[1, 2k - 1] \); that is, \( Y_j \neq U_j = V_j = [2k - j + 1, 2k + j] \), then

\[ V_j' = V_j'' = [2k + 2, 2k + 3] \] \( (U_1 = V_1 = [2k, 2k + 1] \neq Y_1) \).

\[ V_{2k-1} = V_{2k-1} = \{ 1 \} \cup [4, n - 1] \] \( (U_{2k-1} = V_{2k-1} = [2, n - 1] \neq Y_{2k-1}) \).

\[ V_j' = [2k - j - 1, 2k - j] \cup [2k - j + 3, 2k + j]. \text{ } 1 < j < 2k - 1. \]

\[ V_j'' = [2k - j + 1, 2k + j - 2] \cup [2k + j + 1, 2k + j + 2]. \text{ } 1 < j < 2k - 1. \]
(c) If \( j \in E[2k + 2, n - 2] \); that is, \( Y_j \neq U_j = V_j = [j - 2k + 1, 6k - j] \), then
\[
V_j' = [j - 2k - 1, j - 2k] \cup [j - 2k + 3, 6k - j],
\]
\[
V_j'' = ([j - 2k + 1, 6k - j - 2] \cup [6k - j + 1, 6k - j + 2]) \cap [1, n - 1]
\]
(the intersection with \([1, n - 1]\) is only significant for \( j = 2k + 2 \)).

(d) If \( j \in O[2k + 1, n - 1] \); that is, \( Y_j \neq U_j = V_j = [j - 2k, 6k - j - 1] \), then
\[
V_{n-1}' = V_{n-1}'' = [2k - 3, 2k - 2] \quad (U_{n-1} = V_{n-1} = [2k - 1, 2k] \neq Y_{n-1}),
\]
\[
V_{2k+1}' = V_{2k+1}'' = [1, n - 4] \cup \{n - 1\} \quad (U_{2k+1} = V_{2k+1} = [1, n - 2] \neq Y_{2k+1}),
\]
\[
V_j' = [j - 2k - 2, j - 2k - 1] \cup [j - 2k + 2, 6k - j - 1], \quad 2k + 1 < j < n - 1,
\]
\[
V_j'' = [j - 2k, 6k - j - 3] \cup [6k - j, 6k - j + 1], \quad 2k + 1 < j < n - 1.
\]

**Proof.** We will prove the case (b). Other cases may be proved by similar arguments. Note that by Lemma 5.10, since \([2k - j + 1, 2k + j] = V_j = [2k + 2\beta - 2j, 2k + 2\beta - 1] \cap [1, n - 1]\), then \(2\beta = j + 1\); that is, the number of elements in \(Y_j \cap [2k, n - 1]\) is \(j + 1\). Further, since there are at most \(2j\) elements in \(Y_j\) there are at most \(j - 1\) elements in \(Y_j \cap [1, 2k - 1]\).

Suppose \(j = 1\). We know there are two elements in \(Y_1 \cap [2k, n - 1]\) and \(Y_1 \cap [1, 2k - 1]\) is empty. Further \(Y_1 \neq V_1 = [2k, 2k + 1]\). If \(2k + 1 \in Y_1\), then \(c_{2k+1} = 0\) and so by condition 3 of \(\Gamma\), \(c_{2k+1} = 1\) also. But this would mean \(Y_1 = [2k, 2k + 1]\), a contradiction. Hence \(c_{2k+1} = 0\) and \(c_{2k+1} = 0\). It follows that there are two elements in \(Y_1 \cap [2k + 2, n - 1]\) and in fact that \(Y_1 \subseteq [2k + 2, n - 1]\). It is now evident there is an injection \(\rho : Y_1 \rightarrow [2k + 2, 2k + 3]\) such that \(2k \leq \rho(i) \leq i\) for all \(i \in Y_1\). Thus, by Corollary 5.7, \(C_{V_1} = C_{V_1}^{p \in \{2k+2,2k+3\}} = C_{V_1}^{p \in \{2k+2,2k+3\}} = C_{V_1}^{p \in \{2k+2,2k+3\}}, \) for all \(p \in \mathbb{N} \cup \{0\}\).

Now suppose \(j = 2k - 1\). We know there are \(2k - 1\) elements in \(Y_{2k-1} \cap [2k, n - 1]\), so \([2k, n - 1] \subseteq Y_{2k-1}\). Further, there are at most \(j - 1 = 2k - 2\) elements in \(Y_{2k-1} \cap [1, 2k - 1]\). As there are \(2k - 1\) elements in \([1, 2k - 1]\), there is at least one \(i \in [1, 2k - 1]\) such that \(c_{2k-1} = 0\). Note that by property 3 of \(\Gamma\), if \(i \in O[3, 2k - 1]\) (respectively, \(i \in E[2, 2k - 2]\)) then also \(c_{i+1} = 0\) (respectively, \(c_{i+1} \neq 0\)) and \(i \in E[2, 2k - 2]\) (respectively, \(i + 1 \in O[3, 2k - 1]\)). Thus there for to be exactly one \(i\) such that \(c_{2k-1} = 0\), \(i = 1\) and so \(Y_{2k-1} = [2, n - 1] \subseteq V_{2k-1}\) which is a contradiction. It follows that there is at least one \(i' \in O[3, 2k - 1]\) such that \(c_{i'} = 0\); that is, \(Y_{2k-1} \cap [1, 2k - 1] \subseteq [1, i' - 2] \cup [i' + 1, 2k - 1]\) for some \(i' \in O[3, 2k - 1]\). Thus there is an injection \(\rho_1 : Y_{2k-1} \cap [1, 2k - 1] \rightarrow \{1\} \cup [4, 2k - 1] = V_{2k-1} \cap [1, 2k - 1]\) such that \(\rho_1(i'') \leq 2k - 1\) for all \(i'' \in V_{2k-1} \cap [1, 2k - 1]\).

Since \(k \geq 2\) (recall the case \(k = 1\) requires no proof since the permutation of maximum entropy is a cycle when \(k = 1\), \(V_{2k-1} \cap [2k, n - 1] = [2k, n - 1] \subseteq Y_{2k-1} \cap [2k, n - 1]\). Thus there is an injection \(\rho : Y_{2k-1} \rightarrow V_{2k-1} \cap [2k, n - 1]\) such that \(\rho(i'') \leq 2k - 1\) or \(2k \leq \rho(i'') = i''\) for all \(i'' \in V_{2k-1}\). Thus, by Corollary 5.7
\[
C_{V_{2k-1}}^{p} \leq C_{V_{2k-1}}^{p} \leq C_{V_{2k-1}}^{p},
\]
for all \(p \in \mathbb{N} \cup \{0\}\).
Finally, suppose $1 < j < 2k - 1$. We know we have $j+1$ elements in $Y_j \cap [2k, n-1]$ and at most $j-1$ elements in $Y_j \cap [1, 2k-1]$. In the instance where $Y_j \cap [2k, n-1] = [2k, 2k + j] = V_j \cap [2k, n-1] = V_j' \cap [2k, n-1]$ note that $Y_j \cap [2k - j + 1, 2k - 1] \neq [2k - j + 1, 2k - 1]$ since otherwise we would have $Y_j = [2k - j + 1, 2k + 1] = V_j$. Thus there is at least one $i \in [2k - j + 1, 2k - 1]$ such that $c_{i,j} = 0$. Using property 3 of $\Gamma$ and the fact that $2k - j + 1$ is even and a similar argument to the above establishes the existence of at least one $i' \in O[2k - j + 2, 2k - 1]$ such that $Y_j \cap [2k - j + 1, 2k - 1] \subseteq [2k - j + 1, i' - 2] \cup [i' + 1, 2k - 1]$ (note that if $i' = 2k - j + 2$, $[2k - j + 1, i' - 2] = \emptyset$ and if $i' = 2k - 1$, $[i' + 1, 2k - 1] = \emptyset$). Now $V_j' \cap [2k - j + 1, 2k - 1] = [2k - j + 3, 2k - 1]$, which contains the $j - 3$ elements “most centrally spaced”; that is, closest to $2k - 1$ in $[2k - j + 1, 2k - 1]$, while $Y_j \cap [2k - j + 1, 2k - 1]$ contains at most $j - 3$ elements. Also, $V_j'' \cap [1, 2k - j] = [2k - j - 1, 2k - j]$ which contains the two elements “most centrally spaced” in $[1, 2k - j]$. Combining this with the fact that $Y_j \cap [1, 2k - 1]$ has at most $j - 1$ elements, we deduce the existence of an injection $\rho : Y_j \cap [1, 2k - 1] \rightarrow V_j'' \cap [1, 2k - 1]$ such that $\rho(i') \leq \rho(i'') \leq 2k - 1$ for all $i'' \in Y_j \cap [1, 2k - 1]$. As $Y_j \cap [2k, n-1] = V_j' \cap [2k, n-1]$ this is enough to apply Corollary 5.7 and deduce $C_{V_j}^p \leq C_{V_j'}^p$ for all $p \in \mathbb{N} \cup \{0\}$.

In the instance where $Y_j \cap [2k, n-1] \neq [2k, 2k + j]$ we use the fact that $Y_j \cap [2k, n-1]$ has $j + 1$ elements and exploit property 3 of $\Gamma$ in the now familiar manner to establish first the existence of at least one $i' \in O[2k + 1, 2k + j]$ such that $Y_j \cap [2k, 2k + j] \subseteq [2k, i' - 2] \cup [i' + 1, 2k + j]$, and then the fact that $V_j'' \cap [2k, 2k + j] = [2k, 2k + j - 2]$ and $V_j'' \cap [2k + j + 1, n-1] = [2k + j + 1, 2k + j + 2]$ to deduce the existence of an injection $\rho_2 : Y_j \cap [2k, n-1] \rightarrow V_j'' \cap [2k, n-1]$ such that $2k \leq \rho_2(i'') \leq i''$ for all $i'' \in Y_j \cap [2k, n-1]$. Now as $Y_j \cap [1, 2k - 1]$ has at most $j - 1$ elements and $V_j'' \cap [1, 2k - 1] = [2k - j + 1, 2k - 1]$ (the “most central” $j - 1$ element subset of $[1, 2k - 1]$), we see we also have an injection $\rho_1 : Y_j \cap [1, 2k - 1] \rightarrow V_j'' \cap [1, 2k - 1]$ such that $\rho_1(i'') \leq 2k - 1$ for all $i'' \in Y_j \cap [1, 2k - 1]$. Combining the injections and applying Corollary 5.7 gives $C_{V_j}^p \leq C_{V_j''}^p$ for all $p \in \mathbb{N} \cup \{0\}$ as required. \hfill \Box

Observe that we have shown that if $j \in [1, 2k - 1] \cup [2k + 1, n-1]$ and for each $C \in \Gamma$ if $Y_j = S_j(C)$ then there exists $V \in T_j$ such that for all $p \in \mathbb{N} \cup \{0\}$

$$C_{V_j}^p \leq C_{V}^p \quad \text{(Corollary 5.11).}$$

Further to this, if $Y_j \neq U_j = S_j(A)$ but $V_j = U_j$ where $V_j$ is constructed from $Y_j$ as in Lemma 5.10, then for all $p \in \mathbb{N} \cup \{0\}$

$$C_{V_j}^p \leq C_{V_j'}^p \quad \text{or} \quad C_{V_j}^p \leq C_{V_j''}^p,$$

where $V_j'$ and $V_j''$ are defined in Lemma 5.12. These observations may be combined to form the following main corollary used in the proof of the main result.

**Corollary 5.13.** Let $C \in \Gamma$ and let $j \in [1, 2k - 1] \cup [2k + 1, n-1]$, $Y_j = S_j(C)$. $U_j = S_j(A)$, $T_j$ be as defined in Corollary 5.11 and $V_j'$ and $V_j''$ be defined as in Lemma 5.12. Define $T_j'$ by

$$T_j' = (T_j \setminus \{U_j\}) \cup \{V_j', V_j''\}.$$ 

Then there exists $W \in T_j$ such that $C_{V_j}^p \leq C_{W}^p$ for all $p \in \mathbb{N} \cup \{0\}$. Further, if $Y_j \neq U_j$ then there exists $W \in T_j'$ such that $C_{V_j}^p \leq C_{W}^p$ for all $p \in \mathbb{N} \cup \{0\}$. 


**Proof.** Let $V_j$ and $V$ be the sets constructed from $Y_j$ in Lemma 5.10 and Corollary 5.11 respectively and recall our remark following Corollary 5.11 that $V = U_j$ if and only if $V_j = U_j$. By Corollary 5.11 if we wish to find $W \in T_j$ such that $C_{Y_j}^p \leq C_W^p$ for all $p \in \mathbb{N} \cup \{0\}$ we may choose $W$ to be $V$. Further, if $V \neq U_j$ then to find a $W \in T_j'$ such that $C_{Y_j}^p \leq C_W^p$ for all $p \in \mathbb{N} \cup \{0\}$ we may still choose $W$ to be $V$. Finally, if $V = U_j$, and so consequently $V_j = U_j$ but $Y_j \neq U_j$, an appeal to Lemma 5.12 shows that there is at least one choice of $V_j'$ or $V_j''$ for $W$; that is, there is a $W \in T_j'$ such that $C_{Y_j}^p \leq C_W^p$ for all $p \in \mathbb{N} \cup \{0\}$. ◻

The notation used in the above result and in its supporting results Lemmas 5.10, 5.12 and Corollary 5.11 has proved useful in efficiently specifying these results. However as mentioned earlier, we will use an alternative notation for the proofs of our main results. It is clear that the distribution of 1’s in a given column of a matrix $C$ has great bearing on the sum $C_{Y_j}^p$. Furthermore, the proximity of these 1’s to $2k$ is of vital importance. Thus the notation we will use now describes the column with the emphasis on the central element $2k$. For example, for $j \in E[3k + 1, n - 2]$ we have $B^{(j)} = (j - 2k + 1, 6k - j)$ in our previous notation and $B^{(j)} = B^{(n - 2i)} = (2k - 2i + 1, 2k + 2i)$ for $1 \leq i \leq (k - 1)/2$ in the new notation. We now see immediately that the unit elements of $B^{(j)}$ are as close to $2k$ as possible. This fact is not so obvious in the previous notation. With this in mind we respecify $T_j$ and $T_j'$ broken up as they are actually used in our main inductive proofs. Note that we do not need to specify $T_j$ for $j \in E[2, k - 1]$ or $j \in O[3k + 1, n - 1]$ as we need to treat the cases $Y_j = U_j$ quite separately here than elsewhere. (These being the major problem cases in which $X_j \neq U_j$.)

To assist the reader in the cases where the old notation may make the statement of the desired results easier to understand than the new notation used in the proof of the results, we will state the results as we prove them (that is, with the new notation first, and then immediately restate the results in the old notation).

**Representation 5.14.** The following new representation is given for the collection of subsets $T_j$ and $T_j'$ defined earlier in Corollaries 5.11 and 5.13.

(i)

\[ T_1 = \{[2k - 2, 2k - 1], [2k, 2k + 1]\} \]

\[ T_{2i-1} = \{[2k - 2i + 2 + 2\alpha, 2k + 2i - 1 + 2\alpha] : \alpha \in [1, i - 1]\} \]
\[ \cup \{[2k - 2i + 2 - 2\alpha, 2k + 2i - 1 - 2\alpha] : \alpha \in [1, i]\} \]
\[ \cup \{[2k - 2i + 2, 2k + 2i - 1]\}, \text{ for } i \in [2, k/2]. \]

\[ T_1' = \{[2k - 2, 2k - 1], [2k + 2, 2k + 3]\} \]

\[ T_{2i-1}' = (T_{2i-1} \setminus \{[2k - 2i + 2, 2k + 2i - 1]\}) \]
\[ \cup \{[2k - 2i, 2k - 2i + 1] \cup [2k - 2i + 4, 2k + 2i - 1]\} \]
\[ \cup \{[2k - 2i + 2, 2k + 2i - 3] \cup [2k + 2i, 2k + 2i + 1]\}, \text{ for } i \in [2, k/2]. \]
(ii) 
\[ T_{n-2i} = \{ [2k - 2i + 1 + 2\alpha, 2k + 2i + 2\alpha] : \alpha \in [1, i] \} \]
\[ \quad \cup \{ [2k - 2i + 1 - 2\alpha, 2k + 2i - 2\alpha] : \alpha \in [1, i] \} \]
\[ \quad \cup \{ [2k - 2i + 1, 2k + 2i] \}, \text{ for } i \in [1, (k-1)/2]. \]

\[ T'_{n-2i} = (T_{n-2i} \setminus \{ [2k - 2i + 1, 2k + 2i] \}) \]
\[ \quad \cup \{ [2k - 2i - 1, 2k - 2i] \cup [2k - 2i - 3, 2k + 2i] \} \]
\[ \quad \cup \{ [2k - 2i + 1, 2k + 2i - 2] \cup [2k + 2i + 1, 2k + 2i + 2] \}, \]
\[ \quad \text{for } i \in [1, (k-1)/2]. \]

(iii) 
\[ T'_{2i} = \{ [2k - 2i + 2\alpha, 2k + 2i - 1 + 2\alpha] : \alpha \in [1, i] \} \]
\[ \quad \cup \{ [2k - 2i - 2\alpha, 2k + 2i - 1 - 2\alpha] : \alpha \in [1, i] \} \]
\[ \quad \cup \{ [2k - 2i - 2, 2k - 2i - 1] \cup [2k - 2i + 2, 2k + 2i - 1] \} \]
\[ \quad \cup \{ [2k - 2i, 2k + 2i - 3] \cup [2k + 2i, 2k + 2i + 1] \}, \]
\[ \quad \text{for } i \in [1, (k-1)/2]. \]

(iv) 
\[ T'_{n-1} = \{ [2k + 1, 2k + 2], [2k - 3, 2k - 2] \} \]

\[ T'_{n-2i+1} = \{ [2k - 2i + 1 + 2\alpha, 2k + 2i - 2 + 2\alpha] : \alpha \in [1, i] \} \]
\[ \quad \cup \{ [2k - 2i + 1 - 2\alpha, 2k + 2i - 2 - 2\alpha] : \alpha \in [1, i - 1] \} \]
\[ \quad \cup \{ [2k - 2i - 1, 2k - 2i] \cup [2k - 2i + 3, 2k + 2i - 2] \} \]
\[ \quad \cup \{ [2k - 2i + 1, 2k + 2i - 4] \cup [2k + 2i - 1, 2k + 2i] \}, \]
\[ \quad \text{for } i \in [2, k/2]. \]

(v) 
\[ T_{2k+1} = \{ [3, n - 1], [1, n - 2] \} \]

\[ T_{2k+2i-1} = \{ [2i - 1 + 2\alpha, n - 2i + 2\alpha] : \alpha \in [1, i - 1] \} \]
\[ \quad \cup \{ [2i - 1 - 2\alpha, n - 2i - 2\alpha] : \alpha \in [1, i - 1] \} \]
\[ \quad \cup \{ [4i - 1, n - 1] \} \cup \{ [2i - 1, n - 2i] \}, \text{ for } i \in [2, (k + 1)/2]. \]

\[ T'_{2k+1} = \{ [3, n - 1], [1, n - 4] \cup \{ n - 1 \} \} \]

\[ T'_{2k+2i-1} = (T_{2k+2i-1} \setminus \{ [2i - 1, n - 2i] \}) \]
\[ \quad \cup \{ [2i - 3, 2i - 2] \cup [2i + 1, n - 2i] \} \]
\[ \quad \cup \{ [2i - 1, n - 2i - 2] \cup [n - 2i + 1, n - 2i + 2] \}, \]
for \( i \in [2, (k + 1)/2] \).

(vi)

\[ T_{2k+2} = \{ [1, n - 4], [5, n - 1], [3, n - 2] \} \]

\[ T_{2k+2i} = \{ [2i + 1 + 2\alpha, n - 2i + 2\alpha] : \alpha \in [1, i - 1] \} \]

\[ \cup \{ [2i + 1 - 2\alpha, n - 2i - 2\alpha] : \alpha \in [1, i] \} \]

\[ \cup \{ [4i + 1, n - 1] \} \cup \{ [2i + 1, n - 2i] \}, \]

for \( i \in [2, k/2] \).

\[ T'_{2k+2i} = (T_{2k+2i} \setminus \{ [2i + 1, n - 2i] \}) \cup \{ [2i - 1, 2i] \cup [2i + 3, n - 2i] \} \]

\[ \cup \{ [1, n - 1] \cap ([2i + 1, n - 2i - 2] \cup [n - 2i + 1, n - 2i + 2]) \} \]

for \( i \in [1, k/2] \).

(vii)

\[ T_{2k-1} = \{ [1, n - 3], [2, n - 1] \} \]

\[ T_{2k-2i+1} = \{ [2i + 2\alpha, n - 2i + 1 + 2\alpha] : \alpha \in [1, i - 1] \} \]

\[ \cup \{ [2i - 2\alpha, n - 2i + 1 - 2\alpha] : \alpha \in [1, i - 1] \} \]

\[ \cup \{ [1, n - 4i + 1] \} \cup \{ [2i, n - 2i + 1] \}, \]

for \( i \in [2, (k + 1)/2] \).

\[ T'_{2k-1} = \{ [1, n - 3], [1] \cup [4, n - 1] \} \]

\[ T'_{2k-2i+1} = (T_{2k-2i+1} \setminus \{ [2i, n - 2i + 1] \}) \]

\[ \cup \{ [2i - 2, 2i - 1] \cup [2i + 2, n - 2i + 1] \} \]

\[ \cup \{ [2i, n - 2i - 1] \cup [n - 2i + 2, n - 2i + 3] \}, \]

for \( i \in [2, (k + 1)/2] \).

(viii)

\[ T_{2k-2} = \{ [4, n - 1], [1, n - 5], [2, n - 3] \} \]

\[ T_{2k-2i} = \{ [2i + 2\alpha, n - 2i - 1 + 2\alpha] : \alpha \in [1, i] \} \]

\[ \cup \{ [2i - 2\alpha, n - 2i - 1 - 2\alpha] : \alpha \in [1, i - 1] \} \]

\[ \cup \{ [1, n - 4i - 1] \} \cup \{ [2i, n - 2i - 1] \}, \]

for \( i \in [2, k/2] \).

\[ T'_{2k-2i} = (T_{2k-2i} \setminus \{ [2i, n - 2i - 1] \}) \]
\[ U \{[2i - 2, 2i - 1] \cup [2i + 2, n - 2i - 1] \cap [1, n - 1] \} \]
\[ \cup \{[2i, n - 2i - 3] \cup [n - 2i, n - 2i + 1] \}, \]
for \( i \in [1, k/2] \).

6. Two inductive lemmas

In this section we show that the norm \( \|B^p\| \) dominates the norm \( \|C^p\| \), where \( C \in \Gamma \). The arguments we use here are inductive. Because of their repetitive nature, we have devised three algorithms to do the job in a more concise way.

**NOTATION 6.1.** We introduce the following notation for Lemmas 45 and 47. For any matrices \( C \) and \( D \),

\[ C_{[a, b]}^m = \sum_{i=a}^{b} |(C^m)^{(i)}| \]

and

\[ (C^m - D^m)[a, b] = \sum_{i=a}^{b} |(C^m)^{(i)}| - \sum_{i=a}^{b} |(D^m)^{(i)}|. \]

**LEMMA 6.2.** Let \( B = M(\theta_n) \) and \( m \in \mathbb{N} \), then

(i) \( (B^m)^{(2i-1)} \geq (B^m)^{(n-2i+1)} \) for all \( i \in [1, k/2] \)

(ii) \( (B^m)^{(k)} = (B^m)^{(3k)} \)

(iii) \( (B^m)^{(2k+2i-1)} \geq (B^m)^{(2k-2i+1)} \) for all \( i \in [1, k/2] \)

(iv) \( (B^m)^{(n-2i)} \geq (B^m)^{(2i)} \) for all \( i \in [1, (k - 1)/2] \)

(v) \( (B^m)^{(2k+2i)} \geq (B^m)^{(2k-2i)} \) for all \( i \in [1, (k - 1)/2] \)

(vi) \( (B^m)^{(2k-2i+2)} \geq (B^m)^{(2k+2i-1)} \) for all \( i \in [1, (k + 1)/2] \)

(vii) \( (B^m)^{(n-2i-1)} \geq (B^m)^{(n-2i)} \) for all \( i \in [1, (k - 1)/2] \)

(viii) \( (B^m)^{(2k-2i+1)} \geq (B^m)^{(2k+2i)} \) for all \( i \in [1, k/2] \)

(ix) \( (B^m)^{(2i)} \geq (B^m)^{(2i-1)} \) for all \( i \in [1, k/2] \).

**LEMMA 6.3.** Restatement of Lemma 6.2 and Corollary. Let \( B = M(\theta_n) \) and \( m \in \mathbb{N} \), then

1. \( (B^m)^{(k)} = (B^m)^{(3k)} \)

2. For \( j \in O[1, k - 1] \), \( (B^m)^{(n-j)} \leq (B^m)^{(j)} \leq (B^m)^{(j+1)} \)

3. For \( j \in E[2, k - 1] \), \( (B^m)^{(j)} \leq (B^m)^{(n-j)} \leq (B^m)^{(n-j-1)} \)

4. For \( j \in [k + 1, 2k] \), \( (B^m)^{(n-j+1)} \leq (B^m)^{(j)} \leq (B^m)^{(n-j)} \)

5. For \( j \in [1, k - 1] \), \( (B^m)^{(j)} \leq (B^m)^{(n-j-1)} \)

and \( (B^m)^{(n-j)} \leq (B^m)^{(j+1)} \).
(1 is 6.2 (ii), 2 is 6.2 (i) and (ix), 3 is 6.2 (iv) and (vii) and 4 is 6.2 (iii), (vi) and (viii) while 5 is a corollary of 2 and 3.)

Lemmas 6.2 and 6.3 replace the use of cones in earlier results of our main type. Note that if we abbreviate \(|(B^m)^{(j)}|\) to \([j]\) we can summarise a complete ranking of the \(j\)th column sums of \(B^m\) in decreasing order as follows:

\[ [2k] \geq [2k + 1] \geq [2k - 1] \geq [2k + 2] \geq [2k - 2] \geq \cdots \]

\[ \cdots \geq [3k + 2] \geq [k - 2] \geq [3k + 1] \geq [k - 1] \geq [k] = [3k] \]

and

\[ [k] = [3k] \geq [3k + 1] \geq [k - 1] \geq [k - 2] \geq [3k + 2] \geq \cdots \]


for \(k\) odd, and

\[ [3k] = [k] \geq [k - 1] \geq [3k + 1] \geq [3k + 2] \geq [k - 2] \geq \cdots \]


for \(k\) even.

**Proof of Lemma 6.2.** Since \(B \in \Gamma\), (ix) and (vii) follow by Corollary 5.6 and (vi) and (viii) follow by Lemma 5.5. The proof of the remainder is by induction. For any \(p \geq 1\) let Claim (1, p), Claim (2, p), \ldots, Claim (5, p) be as follows:

Claim (1, p) \(|(B^p)^{(2i-1)}| \geq |(B^p)^{(n-2i+1)}|\) for all \(i \in [1, k/2]\).

Claim (2, p) \(|(B^p)^{(k)}| = |(B^p)^{(3k)}|\).

Claim (3, p) \(|(B^p)^{(2k+2i-1)}| \geq |(B^p)^{(2k-2i+1)}|\) for all \(i \in [1, k/2]\).

Claim (4, p) \(|(B^p)^{(n-2i)}| \geq |(B^p)^{(2i)}|\) for all \(i \in [1, (k-1)/2]\).

Claim (5, p) \(|(B^p)^{(2k+2i)}| \geq |(B^p)^{(2k-2i)}|\) for all \(i \in [1, (k-1)/2]\).

For \(p = 1\), \(|B^{(j)}| = |B^{(n-j)}|\) for all \(j \in \{1, \ldots, n-1\}\) hence all claims are true.

We now assume all claims are true for \(p = m\) (if \(r \in [1, 5]\) and \(s \in [1, m]\) we refer to the assumed true Claim (r, s) as inductive hypothesis (r, s)) and aim to show all claims are true for \(p = m + 1\).

**Proof of Claim (1, m+1).** Let \(i \in [1, k/2]\). Then

\[ |(B^{m+1})^{(2i-1)}| - |(B^{m+1})^{(n-2i+1)}| \\
= B_m^{[2k-2i+2, 2k+2i-1]} - B_m^{[2k-2i+2, 3k+2i]} \\
= |(B^{m})^{(2k-2i+2)}| - |(B^{m})^{(2k+2i)}| \\
\geq 0 \]

by (vi) and Corollary 5.6.
Proof of Claim (2, m+1).

\[
\left| (B^{m+1})^{(k)} \right| \geq \left| (B^{m+1})^{(3k)} \right|
\]

\[
= \begin{cases} 
B_{[k+1,3k]}^m - B_{[k,3k-1]}^m, & \text{if } k \text{ is odd} \\
B_{[k,3k-1]}^m - B_{[k+1,3k]}^m, & \text{if } k \text{ is even}
\end{cases}
\]

\[
= \begin{cases} 
\left| (B^{m})^{(3k)} \right| - \left| (B^{m})^{(k)} \right|, & \text{if } k \text{ is odd} \\
\left| (B^{m})^{(k)} \right| - \left| (B^{m})^{(3k)} \right|, & \text{if } k \text{ is even}
\end{cases}
\]

\[\geq 0\]

by inductive hypothesis (2, m).

Proof of Claim (3, m+1). Let \(i \in [1, k/2]\). Then

\[
\left| (B^{m+1})^{(2k+2i-1)} \right| \geq \left| (B^{m+1})^{(2k-2i+1)} \right|
\]

\[
= B_{[2i-1,n-2i]}^m - B_{[2i,n-2i+1]}^m
\]

\[\geq 0\]

by inductive hypothesis (1, m).

Proof of Claim (4, m+1). Let \(i \in [1, (k - 1)/2]\). Then

\[
\left| (B^{m+1})^{(n-2i)} \right| \geq \left| (B^{m+1})^{(2i)} \right|
\]

\[
= B_{[2k-2i+1,2k+2i]}^m - B_{[2k-2i+2,2k+2i+1]}^m
\]

\[
= \left| (B^{m})^{(2k-2i+1)} \right| - \left| (B^{m})^{(2k+2i+1)} \right|
\]

\[\geq 0\]

by (viii) and Corollary 5.6.

Proof of Claim (5, m+1). Let \(i \in [1, (k - 1)/2]\). Then

\[
\left| (B^{m+1})^{(2k+2i)} \right| \geq \left| (B^{m+1})^{(2k-2i)} \right|
\]

\[
= B_{[2i+1,n-2i]}^m - B_{[2i,n-2i-1]}^m
\]

\[\geq 0\]

by inductive hypothesis (4, m).

The next lemma is the key to the proof of the main theorem. The inductive statement is very complex so we give a brief overview of the aim of the various parts of this lemma.

Recall that our goal is to show that \(\|B^p\| \geq \|C^p\|\) for all \(p \in \mathbb{N}\) and for all \(C \in \Gamma\). We have already stated that it is not true that \(\|(B^{(j)}) \geq \|(C^{(j)}\) for all
$j \in [1, n - 1]$, but that it is true for most $j \in [1, n - 1]$: indeed condition (v) says that the statement is true for all $j \in O[1, k - 1] \cup \{k\} \cup [2k, 3k] \cup E[3k + 1, n - 2]$. Furthermore, the statement is also true for $j \in E[2, k - 1] \cup O[3k + 1, n - 1]$ provided $C^{(j)} \neq A^{(j)}$ (conditions (v) (a) and (b)) and for $j \in [k + 1, 2k - 1]$ provided $C^{(j)} \neq B^{(j)}$ (= $A^{(j)}$) or $C^{(n-j)} \neq B^{(n-j)}$ (= $A^{(n-j)}$) (conditions (v) (c) and (d)). The cases excluded in conditions (v) (c) and (d) are dealt with in condition (vi) whilst those excluded in conditions (v) (a) and (b) are dealt with in condition (vii). In both of these cases, the problem is resolved by comparing the sums $|\{B^{(m)}(j)\}| + |\{B^{(m)}(s)\}|$ and $|\{C^{(m)}(j)\}| + |\{C^{(m)}(s)\}|$ for particular columns $s$ of the matrices $B$ and $C$.

Conditions (i) to (iv) are necessary to establish conditions (v) to (vii) and we combine them with Lemma 6.2 and Corollary 5.6 to form the concise result

\[
|\{B^{(m)}(s)\}| \geq |\{C^{(m)}(s')\}|
\]

where $s \in [2, n - 2]$ and $s' \in \{s - 1, n - s + 1\}$ for $s \leq 2k$ or $s' \in \{s + 1, n - s - 1\}$ for $s \geq 2k$.

The various column sum comparisons described in conditions (viii) to (xi) are again used as supporting results for conditions (v) to (vii). Of course, a corollary of condition (xi) is precisely the result we require.

**Lemma 6.4.** Let $C \in \Gamma$ and for any $j \in [1, n - 1]$, let $C^{(j)}$ be denoted by $\langle a, b \rangle^j$. (Note that this does not mean that $\langle C^{(j)} \rangle = [a, b]$.) Then for all $m \in \mathbb{N}$,

(i) $|\{B^{(m)}(2i-1)\}| \geq |\{C^{(m)}(n-2i+1)\}|$ for all $i \in [1, k/2]$

(ii) $|\{B^{(m)}(n-2i)\}| \geq |\{C^{(m)}(2i)\}|$ for all $i \in [1, (k-1)/2]$

(iii) $|\{B^{(m)}(2k+2i-1)\}| \geq |\{C^{(m)}(2k-2i+1)\}|$ for all $i \in \{1, (k+1)/2\}$

(iv) $|\{B^{(m)}(2k+2i)\}| \geq |\{C^{(m)}(2k-2i)\}|$ for all $i \in [1, k/2]$

(v) $|\{B^{(m)}(j)\}| \geq |\{C^{(m)}(j)\}|$ for all $j \in O[1, k-1] \cup \{k\} \cup [2k, 3k] \cup E[3k+1, n-2]$

and for all $j$ such that

(a) $j = 2i$, with $i \in [1, (k-1)/2]$ and $\langle a, b \rangle = \langle 2k - 2i, 2k + 2i - 1 \rangle$

or

(b) $j = n - 2i + 1$, with $i \in [1, k/2]$ and $\langle a, b \rangle = \langle 2k - 2i + 1, 2k + 2i - 2 \rangle$

or

(c) $j = 2k - 2i + 1$, with $i \in [1, k/2]$ and $\langle a, b \rangle = \langle 2k - 2i + 1, 2i - n - 1 \rangle$

or

(d) $j = 2k - 2i$, with $i \in [1, (k-1)/2]$ and $\langle a, b \rangle = \langle 2k - 2i, 2i - n - 1 \rangle$

or

(e) $j = 2k + 2i$, with $i \in [1, (k-1)/2]$ and $\langle a, b \rangle = \langle 2k + 2i, 2i - n - 1 \rangle$

(vi) $|\{B^{(m)}(j)\}| + |\{B^{(m)}(n-j)\}| \geq |\{C^{(m)}(j)\}| + |\{C^{(m)}(n-j)\}|$ for $j$ such that

(a) $j = 2k - 2i + 1$, with $i \in [1, (k+1)/2]$ and $\langle a, b \rangle = \langle 2i, n - 2i + 1 \rangle$

and $\langle a, b \rangle = \langle 2i + 1, n - 2i \rangle$

or

(b) $j = 2k + 2i$, with $i \in [1, k/2]$ and $\langle a, b \rangle = \langle 2i, n - 2i - 1 \rangle$

and $\langle a, b \rangle = \langle 2i + 1, n - 2i \rangle$

(vii) $|\{B^{(m)}(j)\}| + |\{B^{(m)}(s)\}| \geq |\{C^{(m)}(j)\}| + |\{C^{(m)}(s)\}|$ for $j \in E[2, k-1] \cup O[3k+1, n-1]$ and $s \in \{n-j, 2k-j, 2k+j : j \in E[2, k-1]\}$ or $s \in \{n-j, j-2k, 6k-j : j \in O[3k+1, n-1]\}$ such that $C^{(j)}$ and $C^{(s)}$ satisfy one of the following conditions:
(a) \((a, b)^{2i} = (2k - 2i, 2k + 2i - 1)\) and \((a, b)^{n-2i} \neq (2k - 2i + 1, 2k + 2i)\) for \(i \in [1, (k-1)/2]\)

(b) \((a, b)^{2i} = (2k - 2i, 2k + 2i - 1)\) and \((a, b)^{2k-2i} \neq (2i, n-2i-1)\) for \(i \in [1, (k-1)/2]\)

(c) \((a, b)^{2i} = (2k - 2i, 2k + 2i - 1)\) and \((a, b)^{2k+2i} \neq (2i + 1, n-2i)\) for \(i \in [1, (k-1)/2]\)

(d) \((a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)\) and \((a, b)^{2i-1} \neq (2k - 2i + 2, 2k + 2i - 1)\) for \(i \in [1, k/2]\)

(e) \((a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)\) and \((a, b)^{2k-2i+1} \neq (2i, n-2i+1)\) for \(i \in [1, k/2]\)

(f) \((a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)\) and \((a, b)^{2k+2i-1} \neq (2i-1, n-2i)\) for \(i \in [1, k/2]\)

(viii) \(B_{m}^{[j,n-j]} \geq C_{m}^{[j,n-j]}\) for all \(j \in [k, 2k]\)

(ix) \(\left|B^{m}(j)\right| + \left|(B^{m}(2k-j))\right| + \left|(B^{m}(2k+j))\right| + \left|(B^{m}(n-j))\right| \geq \left|(C^{m}(j))\right| + \left|(C^{m}(2k-j))\right| + \left|(C^{m}(2k+j))\right| + \left|(C^{m}(n-j))\right|\) for all \(j \in [1, k-1]\)

(x) \(B_{m}^{[j,2k-j]\cup[2k+j,n-j]} \geq C_{m}^{[j,2k-j]\cup[2k+j,n-j]}\) for all \(j \in [1, k-1]\)

(xi) \(B_{m}^{[j,n-j]} \geq C_{m}^{[j,n-j]}\) for all \(j \in [1, 2k]\)

Restatement of Lemma 6.4.

Let \(C \in \Gamma\) and for any \(j \in [1, n-1]\) let \(Y_{j} = S_{j}(C), U_{j} = S_{j}(A)\) and \(X_{j} = S_{j}(B)\). Then for all \(m \in \mathbb{N}\)

1. \(\left|B^{m}(j)\right| \geq \left|(C^{m}(n-j))\right|\) for all \(j \in O(1, k-1) \cup E[3k+1, n-2] \cup [2k+1, 3k]\).

2. If \(j \in [1, n-1]\) then \(\left|B^{m}(j)\right| \geq \left|(C^{m}(j))\right|\) unless \(j \in E[2, k-1] \cup O[3k+1, n-1]\) and \(Y_{j} = U_{j}\), or unless \(j \in [k+1, 2k-1]\) and \(Y_{j} = U_{j} = X_{j}\) and \(Y_{n-j} = U_{n-j} = X_{n-j}\).

3. \(\left|B^{m}(j)\right| + \left|(B^{m})(n-j)\right| \geq \left|(C^{m}(j))\right| + \left|(C^{m}(n-j))\right|\) for all \(j \in [1, 2k-1]\)

4. \(\left|B^{m}(j)\right| + \left|(B^{m})(n-j)\right| \geq \left|(C^{m}(j))\right| + \left|(C^{m}(n-j))\right|\) for
5. \(B_{m}^{[j,n-j]} \geq C_{m}^{[j,n-j]}\) for all \(j \in [k, 2k].\)
6. \(\left|B^{m}(j)\right| + \left|(B^{m})(2k-j)\right| + \left|(B^{m})(2k+j)\right| + \left|(B^{m})(n-j)\right| \geq \left|(C^{m}(j))\right| + \left|(C^{m}(2k-j))\right| + \left|(C^{m}(2k+j))\right| + \left|(C^{m}(n-j))\right|\) for all \(j \in [1, k-1].\)

\(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24.\)

Note that item 8 with \(j = 1\) shows \(B\) dominates \(C\), which is what we are really after. We reemphasize that the use of the notation \((a, b)^{j}\) as an alternative for \(C^{(j)}\) in Lemma 6.4 does not mean that \(Y_{j}\) is of the form \([a, b]\) but rather that \(a\) is the smallest element of \([1, n-1]\) for which \(c_{ij} = 1\) and \(b\) is the largest element of \([1, n-1]\) for which \(c_{ij} = 1\).
PROOF. Let column $B^{(j)}$ be denoted by $(x, y)^{j}$ throughout with $X_{j} = \{i \in [1, n - 1] : b_{i, j} = 1\}$ and $Y_{j} = \{i \in [1, n - 1] : c_{i, j} = 1\}$. The proof is by induction. For any $p \geq 1$ let Claim (1, p), Claim (2, p), ..., Claim (11, p) be as follows:

Claim (1, p) \[ |(B^{p-1})^{(2i-1)}| \geq |(C^{p-1})^{(n-2i+1)}| \] for all $i \in [1, k/2]$.  
Claim (2, p) \[ |(B^{p-1})^{(n-2i)}| \geq |(C^{p-1})^{(2i)}| \] for all $i \in [1, (k - 1)/2]$.  
Claim (3, p) \[ |(B^{p-1})^{(2k+2i-1)}| \geq |(C^{p-1})^{(2k-2i+1)}| \] for all $i \in [1, (k + 1)/2]$.  
Claim (4, p) \[ |(B^{p-1})^{(2k+2i)}| \geq |(C^{p-1})^{(2k-2i)}| \] for all $i \in [1, k/2]$.  
Claim (5, p) \[ |(B^{p})^{(j)}| \geq |(C^{p})^{(j)}| \] for all $j \in O[1, k - 1] \cup \{k\} \cup [2k, 3k] \cup E[3k + 1, n - 2]$ and for all $j$ such that  

(a) $j = 2i$, with $i \in [1, (k - 1)/2]$ and $(a, b)^{2i} \neq (2k - 2i, 2k + 2i - 1)$  
or (b) $j = n - 2i + 1$, with $i \in [1, k/2]$ and  
\[ (a, b)^{n-2i+1} \neq (2k - 2i + 1, 2k + 2i - 2) \]  
or (c) $j = 2k - 2i + 1$, with $i \in [1, k/2]$ and $(a, b)^{2k-2i+1} \neq (2i, n - 2i + 1)$  
or (d) $j = 2k - 2i$, with $i \in [1, (k - 1)/2]$ and $(a, b)^{2k-2i} \neq (2i, n - 2i - 1)$ or  
\[ (a, b)^{2k+2i} \neq (2i + 1, n - 2i). \]  
Claim (6, p) \[ |(B^{p})^{(j)}| + |(B^{p})^{(n-j)}| \geq |(C^{p})^{(j)}| + |(C^{p})^{(n-j)}| \] for $j$ such that  

(a) $j = 2k - 2i + 1$, with $i \in [1, (k + 1)/2]$ and $(a, b)^{2k-2i+1} = (2i, n - 2i + 1)$ and $(a, b)^{n-j} = (a, b)^{2k+2i-1} = (2i - 1, n - 2i)$  
or (b) $j = 2k - 2i$, with $i \in [1, k/2]$ and $(a, b)^{2k-2i} = (2i, n - 2i - 1)$ and  
\[ (a, b)^{n-j} = (a, b)^{2k+2i} = (2i + 1, n - 2i). \]  
Claim (7, p) \[ |(B^{p})^{(j)}| + |(B^{p})^{(s)}| \geq |(C^{p})^{(j)}| + |(C^{p})^{(s)}| \] for $j \in E[2, k - 1] \cup O[3k + 1, n - 1]$ such that $C^{(j)}$ and $C^{(s)}$ satisfy one of the following conditions:  

(a) $(a, b)^{2i} = (2k - 2i, 2k + 2i - 1)$ and $(a, b)^{n-2i} \neq (2k - 2i + 1, 2k + 2i)$ for $i \in [1, (k - 1)/2]$  
or (b) $(a, b)^{2i} = (2k - 2i, 2k + 2i - 1)$ and $(a, b)^{2k-2i} \neq (2i, n - 2i - 1)$ for $i \in [1, (k - 1)/2]$  
or (c) $(a, b)^{2i} = (2k - 2i, 2k + 2i - 1)$ and $(a, b)^{2k+2i} \neq (2i + 1, n - 2i)$ for  
\[ (a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2) \]  
or (d) $(a, b)^{2i-1} = (2k - 2i + 1, 2k + 2i - 2)$ and $(a, b)^{2k+2i} \neq (2i + 1, n - 2i)$ for $i \in [1, k/2]$  
or (e) $(a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)$ and $(a, b)^{2k-2i+1} \neq 2k - 2i + 1$ for $i \in [1, k/2]$  
or (f) $(a, b)^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)$ and $(a, b)^{2k+2i-1} \neq 2k - 2i + 1$.
\((2i - 1, n - 2i)\) for \(i \in [1, k/2]\).

Claim (8, p) \(B^p_{[j, n-j]} \geq C^p_{[j, n-j]}\) for all \(j \in [k, 2k]\).

Claim (9, p) \(|(B^p)^{(j)}| + |(B^p)^{(2k-j)}| + |(B^p)^{(2k+j)}| + |(B^p)^{(n-j)}| \geq |(C^p)^{(j)}| + |(C^p)^{(2k-j)}| + |(C^p)^{(2k+j)}| + |(C^p)^{(n-j)}|\) for all \(j \in [1, k-1]\).

Claim (10, p) \(B^p_{[j, 2k-j] \cup [2k+j, n-j]} \geq C^p_{[j, 2k-j] \cup [2k+j, n-j]}\) for all \(j \in [1, k-1]\).

Claim (11, p) \(B^p_{[j, n-j]} \geq C^p_{[j, n-j]}\) for all \(j \in [1, 2k]\).

For \(p = 1\), \(|B^{(j)}| \geq |C^{(j)}|, |B^{(n-j)}| \geq |C^{(j)}|\) and \(1 = |I^{(n-j)}| = |(B^0)^{(n-j)}| \geq |(C^0)^{(j)}| = |I^{(j)}| = 1\) for all \(j \in [1, n-1]\) by Corollaries 3.3 and 4.7. (Here \(I\) is the identity matrix of the appropriate dimension.) Hence all claims are true for \(p = 1\).

We now assume all claims are true for \(p = 1\) to \(p = m\) (if \(r \in [1, 11]\) and \(s \in [1, m]\) we refer to the assumed true Claim (r, s) as inductive hypothesis (r, s)) and aim to show all claims are true for \(p = m + 1\).

**Proof of Claim (1, m+1).** To show that \(|(B^m)^{(2i-1)}| \geq |(C^m)^{(n-2i+1)}|\) for all \(i \in [1, k/2]\) we consider two cases:

(a) \(\langle a, b \rangle^{n-2i+1} \neq (2k - 2i + 1, 2k + 2i - 2)\);

(b) \(\langle a, b \rangle^{n-2i+1} = (2k - 2i + 1, 2k + 2i - 2)\).

**Case (a).** By inductive hypothesis (5, m)(b) and Lemma 6.2 we have

\[|{(C^m)^{(n-2i+1)}}| \leq |{(B^m)^{(n-2i+1)}}| \leq |{(B^m)^{(2i-1)}}|.\]

**Case (b).**

\[
\begin{align*}
|{(B^m)^{(2i-1)}}| - |{(C^m)^{(n-2i+1)}}| &= B^{m-1}_{[2k-2i+2, 2k+2i-1]} - C^{m-1}_{[2k-2i+1, 2k+2i-2]} \\
&= (B^{m-1} - C^{m-1}) \cdot 2k - 2i + 2, 2k + 2i - 2 \\
&+ |{(B^m)^{(2k+2i-1)}}| - |{(C^m)^{(2k-2i+1)}}| \\
&\geq 0,
\end{align*}
\]

by inductive hypotheses (11, m-1) and (3, m).

**Proof of Claim (2, m+1).** To show that \(|(B^m)^{(n-2i)}| \geq |(C^m)^{(2i)}|\) for all \(i \in [1, (k-1)/2]\) we consider two cases:

(a) \(\langle a, b \rangle^{2i} \neq (2k - 2i, 2k + 2i - 1)\);

(b) \(\langle a, b \rangle^{2i} = (2k - 2i, 2k + 2i - 1)\).

**Case (a).** By inductive hypothesis (5, m)(a) and Lemma 6.2 we have

\[|{(C^m)^{(2i)}}| \leq |{(B^m)^{(2i)}}| \leq |{(B^m)^{(n-2i)}}|.\]
Case (b).

\[
\left| (B^m)^{(n-2i)} \right| - \left| (C^m)^{(2i)} \right| \\
= B^m_{[2k-2i+1, 2k+2i]} - C^m_{[2k-2i, 2k+2i-1]} \\
= (B^m_{-1} - C^m_{-1}) [2k - 2i + 1, 2k + 2i - 1] \\
+ \left| (B^m_{-1})^{(2k+2i)} \right| - \left| (C^m_{-1})^{(2k-2i)} \right| \\
\geq 0,
\]

by inductive hypotheses \((11, m-1)\) and \((4, m)\).

Proof of Claim \((3, m+1)\). To show that \(\left| (B^m)^{(2k+2i-1)} \right| \geq \left| (C^m)^{(2k-2i+1)} \right|\) for all \(i \in [1, (k+1)/2]\) we consider two cases:
- (a) \(\langle a, b \rangle^{2k-2i+1} \neq \langle 2i, n - 2i + 1 \rangle\);
- (b) \(\langle a, b \rangle^{2k-2i+1} = \langle 2i, n - 2i + 1 \rangle\).

Case (a). By inductive hypothesis \((5, m)(c)\) and Lemma 6.2 we have

\[
\left| (C^m)^{(2k-2i+1)} \right| \leq \left| (B^m)^{(2k-2i+1)} \right| \leq \left| (B^m)^{(2k+2i-1)} \right|.
\]

Case (b).

\[
\left| (B^m)^{(2k+2i-1)} \right| - \left| (C^m)^{(2k-2i+1)} \right| \\
= B^m_{[2i-1, n-2i]} - C^m_{[2i, n-2i+1]} \\
= (B^m_{-1} - C^m_{-1}) [2i, n - 2i] + \left| (B^m_{-1})^{(2i-1)} \right| - \left| (C^m_{-1})^{(n-2i+1)} \right| \\
\geq 0,
\]

by inductive hypotheses \((11, m-1)\) and \((1, m)\) provided \(i \neq (k+1)/2\). If \(i = (k+1)/2\) (whence \(i\) is odd), then

\[
\left| (B^m_{-1})^{(2k+2i-1)} \right| - \left| (C^m_{-1})^{(2k-2i+1)} \right| = \left| (B^m_{-1})^{(3k)} \right| - \left| (C^m_{-1})^{(k)} \right| \\
= \left| (B^m_{-1})^{(k)} \right| - \left| (C^m_{-1})^{(k)} \right| \\
\geq 0,
\]

by Lemma 6.2 and inductive hypothesis \((5, m-1)\).

Proof of Claim \((4, m+1)\). To show that \(\left| (B^m)^{(2k+2i)} \right| \geq \left| (C^m)^{(2k-2i)} \right|\) for all \(i \in [1, k/2]\) we consider two cases:
- (a) \(\langle a, b \rangle^{2k-2i} \neq \langle 2i, n - 2i - 1 \rangle\);
- (b) \(\langle a, b \rangle^{2k-2i} = \langle 2i, n - 2i - 1 \rangle\).

Case (a). By inductive hypothesis \((5, m)(d)\) and Lemma 6.2 we have

\[
\left| (C^m)^{(2k-2i)} \right| \leq \left| (B^m)^{(2k-2i)} \right| \leq \left| (B^m)^{(2k+2i)} \right|.
\]
Case (b).

\[
\left| (B^m)^{(2k+2i)} \right| - \left| (C^m)^{(2k-2i)} \right| \\
= B^{m-1}_{[2i+1, n-2i]} - C^{m-1}_{[2i, n-2i-1]} \\
= (B^{m-1} - C^{m-1}) \left[ 2i + 1, n - 2i - 1 \right] + \left| (B^{m-1})^{(n-2i)} \right| - \left| (C^{m-1})^{(2i)} \right| \\
\geq 0,
\]

by inductive hypotheses (11, m−1) and (2, m) provided \( i \neq k/2 \). If \( i = k/2 \) (whence \( i \) is even), then

\[
\left| (B^{m-1})^{(n-2i)} \right| - \left| (C^{m-1})^{(2i)} \right| = \left| (B^{m-1})^{(3k)} \right| - \left| (C^{m-1})^{(k)} \right| \\
\geq 0,
\]

by inductive hypothesis (4, m).

Note that as a consequence of Claims (1, m+1), (2, m+1), (3, m+1), (4, m+1), inductive hypothesis (5, m) and Lemma 6.2 and Corollary 5.6, we have

**Corollary (6.5, M).** For any \( j \in [2, 2k] \)

\[
\left| (B^m)^{(j)} \right| \geq \left| (C^m)^{(j-1)} \right| \text{ and } \left| (B^m)^{(j)} \right| \leq \left| (C^m)^{(n-j+1)} \right|
\]

and

\[
\left| (B^m)^{(n-j)} \right| \geq \left| (C^m)^{(n-j+1)} \right| \text{ and } \left| (B^m)^{(n-j)} \right| \leq \left| (C^m)^{(j-1)} \right|
\]

**Proof.** To see this

(i) Let \( i \in [2, k/2] \). Then

\[
\left| (B^m)^{(2i-1)} \right| \geq \left| (B^m)^{(n-2i+1)} \right| \quad \text{(by Lemma 6.2)} \\
\geq \left| (B^m)^{(n-2i+2)} \right| \quad \text{(by Corollary 5.6)} \\
\geq \left\{ \begin{array}{l}
\left| (C^m)^{(n-2i+2)} \right| \quad \text{(by ind. hyp. (5, m)) or} \\
\left| (C^m)^{(2i-2)} \right| \quad \text{(by Claim (2, m+1))}.
\end{array} \right.
\]

(ii) Let \( i \in [1, (k-1)/2] \). Then

\[
\left| (B^m)^{(n-2i)} \right| \geq \left| (B^m)^{(2i)} \right| \quad \text{(by Lemma 6.2)} \\
\geq \left| (B^m)^{(2i-1)} \right| \quad \text{(by Corollary 5.6)} \\
\geq \left\{ \begin{array}{l}
\left| (C^m)^{(2i-1)} \right| \quad \text{(by ind. hyp. (5, m)) or} \\
\left| (C^m)^{(n-2i+1)} \right| \quad \text{(by Claim (1, m+1))}.
\end{array} \right.
\]

(iii) Let \( i \in [1, k/2] \). Then

\[
\left| (B^m)^{(2k+2i-1)} \right| \geq \left| (B^m)^{(2k-2i+1)} \right| \quad \text{(by Lemma 6.2)} \\
\geq \left| (B^m)^{(2k+2i)} \right| \quad \text{(by Lemma 6.2)} \\
\geq \left\{ \begin{array}{l}
\left| (C^m)^{(2k+2i)} \right| \quad \text{(by ind. hyp. (5, m)) or} \\
\left| (C^m)^{(2k-2i)} \right| \quad \text{(by Claim (4, m+1))}.
\end{array} \right.
\]
(iv) Let \( i \in [1,(k-1)/2] \). Then
\[
\left| \left( B^m \right)^{(2k+2i)} \right| \geq \left| \left( B^m \right)^{(2k-2i)} \right| \quad \text{(by Lemma 6.2)}
\]
\[
\geq \left| \left( B^m \right)^{(2k+2i+1)} \right| \quad \text{(by Lemma 6.2)}
\]
\[
\geq \begin{cases} 
\left| \left( C^m \right)^{(2k+2i+1)} \right| & \text{(by ind. hyp. (5, m)) or} \\
\left| \left( C^m \right)^{(2k-2i-1)} \right| & \text{(by Claim (3, m+1)).}
\end{cases}
\]

(v) Let \( j = 2k \). Then
\[
\left| \left( B^m \right)^{(2k)} \right| \geq \left| \left( B^m \right)^{(2k+1)} \right| \quad \text{(by Corollary 5.6)}
\]
\[
\geq \begin{cases} 
\left| \left( C^m \right)^{(2k-1)} \right| & \text{(by Claim (3, m+1)) or} \\
\left| \left( C^m \right)^{(2k+1)} \right| & \text{(by ind. hyp. (5, m)).}
\end{cases}
\]

(vi) Let \( j = k \). Then
\[
\left| \left( B^m \right)^{(k)} \right| = \left| \left( B^m \right)^{(3k)} \right|
\]
\[
\geq \left| \left( C^m \right)^{(3k)} \right| \quad \text{(by ind. hyp. (5, m))}
\]
\[
\geq \left| \left( C^m \right)^{(3k+1)} \right| \quad \text{(by Corollary 5.6)}
\]

and
\[
\left| \left( B^m \right)^{(3k)} \right| = \left| \left( B^m \right)^{(k)} \right|
\]
\[
\geq \left| \left( C^m \right)^{(k)} \right| \quad \text{(by ind. hyp. (5, m))}
\]
\[
\geq \left| \left( C^m \right)^{(k-1)} \right| \quad \text{(by Corollary 5.6)}.
\]

This completes the proof of Corollary (6.5, m).

Proof of Claims (5, m+1) and (7, m+1). We first remark that it is easy to see that Claim (5, p) is equivalent to the following:

Claim (5', p) \( \left| \left( B^p \right)^{(j)} \right| \geq \left| \left( C^p \right)^{(j)} \right| \) for all \( j \in [1, n-1] \)

such that

(a) \( j \notin [k+1, 2k-1] \) and \( \langle a, b \rangle^j = \langle x, y \rangle^j \)

or (b) \( j = 2i - 1 \), with \( i \in [1, k/2] \) and \( \langle a, b \rangle^{2i-1} \neq \langle x, y \rangle^{2i-1} = \langle 2k - 2i + 2, 2k + 2i - 1 \rangle \)

or (c) \( j = n - 2i \), with \( j \in [1, (k-1)/2] \) and \( \langle a, b \rangle^{n-2i} \neq \langle x, y \rangle^{n-2i} = \langle 2k - 2i + 1, 2k + 2i \rangle \)

or (d) \( j = 2i \), with \( i \in [1, (k-1)/2] \) and \( \langle a, b \rangle^{2i} \neq \langle x, y \rangle^{2i} = \langle 2k - 2i + 2, 2k + 2i \rangle \).
2k + 2i + 1) and \(\langle a, b \rangle^{2i} \neq \langle 2k - 2i, 2k + 2i - 1 \rangle\)

or (e) \(j = n - 2i + 1\), with \(i \in [1, k/2]\) and \(\langle a, b \rangle^{n-2i+1} \neq \langle (x, y)^{n-2i+1}, \rangle \langle 2k - 2i + 3, 2k + 2i \rangle\) and \(\langle a, b \rangle^{n-2i+1} \neq \langle 2k - 2i + 1, 2k + 2i - 2 \rangle\)

or (f) \(j = 2k + 2i - 1\), with \(i \in [1, (k+1)/2]\) and \(\langle a, b \rangle^{2k+2i-1} \neq \langle (x, y)^{2k+2i-1}, \rangle \langle 2i - 1, n - 2i \rangle\)

or (g) \(j = 2k + 2i\), with \(i \in [1, k/2]\) and \(\langle a, b \rangle^{2k+2i} \neq \langle (x, y)^{2k+2i}, \rangle \langle 2i + 1, n - 2i \rangle\)

or (h) \(j = 2k - 2i + 1\), with \(i \in [1, (k+1)/2]\) and \(\langle a, b \rangle^{2k-2i+1} \neq \langle (x, y)^{2k-2i+1}, \rangle \langle 2i, n - 2i + 1 \rangle\)

or (h') \(j = 2k - 2i + 1\), with \(i \in [1, k/2]\) and \(\langle a, b \rangle^{2k-2i+1} = \langle 2i, n - 2i + 1 \rangle\) and \(\langle a, b \rangle^{2k+2i-1} \neq \langle 2i - 1, n - 2i \rangle\)

or (i) \(j = 2k - 2i\), with \(i \in [1, k/2]\) and \(\langle a, b \rangle^{2k-2i} \neq \langle (x, y)^{2k-2i}, \rangle \langle 2i, n - 2i - 1 \rangle\)

or (i') \(j = 2k - 2i\), with \(i \in [1, (k-1)/2]\) and \(\langle a, b \rangle^{2k-2i} = \langle 2i, n - 2i - 1 \rangle\)

and \(\langle a, b \rangle^{2k+2i} \neq \langle 2i + 1, n - 2i \rangle\).

Thus we will establish Claim \((5', m+1)\).

In fact we will only deal directly with the cases (a), (h'), and (i') where specifically we know \(\langle a, b \rangle^{j} = \langle x, y \rangle^{j}\). The multitudinous cases remaining to be proved in Claim \((5', m+1)\), and the even greater number of cases remaining in Claim \((7, m+1)\) will be handled using Corollary 5.13 and Lemma \((6.7, m)\) below, for which, in turn, we devise a number of algorithms to generate proofs in all possible cases.

**Direct proofs of (a), (h') and (i') of Claim \((5', m+1)\).**

(a) Let \(j \not\in [k+1, 2k-1]\) be such that \(\langle a, b \rangle^{j} = \langle x, y \rangle^{j}\). For \(j = 2i - 1\) and \(1 \leq i \leq k/2\) (that is, \(j \in O[1, k-1]\)) we have \(\langle a, b \rangle^{j} = \langle x, y \rangle^{j} = \langle 2k-2i + 2, 2k+2i-1 \rangle\), and so

\[
\left|\left(B^{m+1}\right)^{(j)}\right| - \left|\left(C^{m+1}\right)^{(j)}\right| = \left(B^{m} - C^{m}\right)[2k - 2i + 2, 2k + 2i - 2]
+ \left|\left(B^{m}\right)^{(2k+2i-1)}\right| - \left|\left(C^{m}\right)^{(2k+2i-1)}\right|
\geq 0,
\]

by inductive hypotheses \((11, m)\) and \((5, m)\).

Similar proofs that \(\left|\left(B^{m+1}\right)^{(j)}\right| - \left|\left(C^{m+1}\right)^{(j)}\right| \geq 0\) for identical reasons hold in the cases \(j = 2i, 1 \leq i \leq (k-1)/2; j = k, k\) odd; \(j = k, k\) even; \(j = 2k+2i-1, 1 \leq i \leq (k+1)/2; j = 2k + 2i, 1 \leq i \leq k/2; j = n - 2i, 1 \leq i \leq (k-1)/2\) and \(j = n - 2i + 1, 1 \leq i \leq k/2\). This covers every case except for the case \(j = 2k\) which is even simpler as it merely relies on inductive hypothesis \((11, m)\).

(h') Let \(j = 2k - 2i + 1\), with \(i \in [1, k/2]\) and \(\langle a, b \rangle^{2k-2i+1} = \langle (x, y)^{2k-2i+1}, \rangle \langle 2i, n - 2i + 1 \rangle\) but \(\langle a, b \rangle^{2k+2i-1} \neq \langle (x, y)^{2k+2i-1}, \rangle \langle 2i - 1, n - 2i \rangle\). Then

\[
\left|\left(B^{m+1}\right)^{(j)}\right| - \left|\left(C^{m+1}\right)^{(j)}\right| = \left(B^{m} - C^{m}\right)[2i, n - 2i + 1].
\]
Case (1). When \((a, b)^{n-2i+1} \neq \langle 2k - 2i + 1, 2k + 2i - 2 \rangle\).

\[
\left| (B^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(j)} \right| = (B^m - C^m)[2i, n - 2i] + (B^m - C^m)[2k - 2i + 1, 2k + 2i - 2] + (B^m - C^m)(n-2i+1) - (C^m)(n-2i+1) \geq 0
\]

by inductive hypotheses (11, m) and (5', m}(e).

Case (2). When \((a, b)^{n-2i+1} = \langle 2k - 2i + 1, 2k + 2i - 2 \rangle\).

\[
\left| (B^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(j)} \right| = (B^m - C^m)([2i, 2k - 2i] \cup [2k + 2i, n - 2i]) + (B^m - C^m)[2k - 2i + 2, 2k + 2i - 2] + (B^m - C^m)(n-2i+1) + (B^m)(2k+2i-1) - (C^m)(n-2i+1) + (C^m)(2k+2i-1) - (C^m)(2k-2i+1) \geq 0
\]

by inductive hypotheses (10, m), (11, m), (7, m}(f) and (5', m}(h')

(i') Let \(j = 2k - 2i\) with \(i \in [1, (k-1)/2]\) and \((a, b)^{2k-2i} = \langle x, y \rangle^{2k-2i} = \langle 2i, n - 2i - 1 \rangle\) but \((a, b)^{2k+2i} \neq \langle (x, y)^{2k+2i} = \langle 2i + 1, n - 2i \rangle\). Then

\[
\left| (B^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(j)} \right| = (B^m - C^m)[2i, n - 2i - 1].
\]

Case (1). When \((a, b)^{2i} \neq \langle 2k - 2i, 2k + 2i - 1 \rangle\).

\[
\left| (B^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(j)} \right| = (B^m - C^m)[2i + 1, n - 2i - 1] + (B^m)(2i) - (C^m)(2i) \geq 0
\]

by inductive hypotheses (11, m) and (5', m}(d).

Case (2). When \((a, b)^{2i} = \langle 2k - 2i, 2k + 2i - 1 \rangle\).

\[
\left| (B^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(j)} \right| = (B^m - C^m)([2i + 1, 2k - 2i - 1] \cup [2k + 2i + 1, n - 2i - 1]) + (B^m - C^m)[2k - 2i + 1, 2k + 2i - 1] + (B^m)(2i) + (B^m)(2k+2i) - (C^m)(2i) + (C^m)(2k+2i) + (C^m)(2k-2i) - (C^m)(2k-2i) \geq 0
\]

by inductive hypotheses (10, m), (11, m), (7, m}(c) and (5', m}(i')(h').

We now present a definition and an accompanying lemma designed to greatly simplify the proof of the remainder of Claim (5', m+1) and of Claim (7, m+1).
DEFINITION 6.6. For any non-empty subset $W$ of $[1, n - 1]$ let $J(W) = \{ j \in [0, 2k - 1] : [2k - j, 2k + j] \subseteq W \}$, then the centre cut $c(W)$ of $W$ is given by

$$c(W) = \begin{cases} 
\emptyset, & \text{if } J = \emptyset \\
[2k - j, 2k + j] & \text{where } j = \max J(W), \text{ if } J \neq \emptyset.
\end{cases}$$

Further, if $J = \emptyset$, which is true if and only if $2k \notin W$, then the centre span $s(W)$ of $W$ is given by

$$s(W) = W \cap \{2k - j, 2k + j\} \text{ where } j = 1 + \max J([1, n - 1] \setminus W).$$

Finally, if $J \neq \emptyset$, $s(W) = \{2k\}$. (So $c(W)$ is the "largest" subinterval of $W$ which is symmetric about $2k$, and $s(W)$ is the set of elements of $W$ which are "closest" to $2k$.) We also make use of the set $t(W)$ given by $t(W) = s(W) \cup \{n-j : j \in s(W)\}$.

LEMMA (6.7, M).

(I) $\left|B_{\mathcal{W}j}^{m+1}(j)\right| = B_{\mathcal{W}j}^m \geq C_{\mathcal{W}}^m$ for

(i) $j = 2i - 1$, with $i \in [1, k/2]$ and $W \in (T_{2i-1} \setminus \{X_{2i-1}\}) \cup \{[2k - 2i + 1, 2k + 2i - 2]\}$

(ii) $j = n - 2i$, with $i \in [1, (k - 1)/2]$ and $W \in (T_{n-2i} \setminus \{X_{n-2i}\}) \cup \{[2k - 2i, 2k + 2i - 1]\}$

(iii) $j = 2i$, with $i \in [1, (k - 1)/2]$ and $W \in T_{2i} \setminus \{X_{2i}\}$

(iv) $j = n - 2i + 1$, with $i \in [1, k/2]$ and $W \in T_{n-2i+1} \setminus \{X_{n-2i+1}\}$

(v) $j = 2k + 2i - 1$, with $i \in [1, (k + 1)/2]$ and $W \in T_{2k+2i-1} \setminus \{X_{2k+2i-1}\}$

(vi) $j = 2k + 2i$, with $i \in [1, k/2]$ and $W \in T_{2k+2i} \setminus \{X_{2k+2i}\}$

(vii) $j = 2k - 2i + 1$, with $i \in [1, (k + 1)/2]$ and $W \in T_{2k-2i+1}$

(viii) $j = 2k - 2i$, with $i \in [1, k/2]$ and $W \in T_{2k-2i}$.

(II) $B_{\mathcal{W}j}^m \geq C_{\mathcal{W}}^m$ for

(i) $Z = X_{2i}$, with $i \in [1, (k - 1)/2]$ and $W \in T_{n-2i}^i$

(ii) $Z = X_{n-2i+1}$, with $i \in [1, k/2]$ and $W \in T_{2i-1}$

(iii) $Z = [2i + 1, n - 2i + 1]$, with $i \in [1, (k - 1)/2]$ and $W \in T_{2k+2i-1} \setminus \{[2i - 1, n - 2i - 2], [2i - 1, 2i] \cup [2i + 1, n - 2i - 2], [2i - 1, n - 2i - 1] \cap ([2i + 1, n - 2i - 2] \cup [n - 2i + 1, n - 2i + 2])\}$

(iv) $Z = [2i - 1, n - 2i - 2]$, with $i \in [1, (k - 1)/2]$ and $W \in T_{2k-2i} \setminus \{[2i + 1, 2i - 2], [2i, n - 2i - 3] \cup [n - 2i, n - 2i + 1]\}$

(v) $Z = [2i + 1, n - 2i + 2]$, with $i \in [2, k/2]$ and $W \in T_{2k-2i+1} \setminus \{[2i - 1, n - 2i - 1], [2i - 1, 2i - 2] \cup [2i + 1, n - 2i + 1]\}$

(vi) $Z = [2i - 2, n - 2i - 1]$, with $i \in [2, k/2]$ and $W \in T_{2k+2i-1} \setminus \{[2i + 1, n - 2i + 2], [2i - 1, n - 2i - 2] \cup [n - 2i + 1, n - 2i + 2], [2i - 3, 2i - 2] \cup [2i + 1, n - 2i - 2]\}$

\textbf{Proof.} In each case we wish to show $B_{\mathcal{W}j}^m \geq C_{\mathcal{W}}^m$; that is, $B_{\mathcal{W}j}^m - C_{\mathcal{W}}^m \geq 0$, for specified subsets of $Z$ and $W$ of $[1, n - 1]$.

The proof of this result, which is central to the paper, is extremely repetitive. In fact it is possible to set up a single algorithm which produces a proof of every case of the result, but the algorithm itself is overly complicated in a large number of the cases. A better solution is to use two much simpler algorithms, the first for handling the vast majority of cases, and the second for handling the remaining cases.
Algorithm 6.8. This produces a proof of all the above cases with the exception of the following:

I(i) when \( W = [2k - 2i + 1, 2k + 2i - 2] \)
I(ii) when \( W = [2k - 2i, 2k + 2i - 1] \)
I(iii) when \( W = [2k - 2i, 2k + 2i - 3] \cup [2k + 2i, 2k + 2i + 1] \)
I(iv) when \( W = [2k - 2i + 1, 2k + 2i - 4] \cup [2k + 2i - 1, 2k + 2i] \) and \( i > 1 \)
I(i) when \( W = [2k - 2i - 1, 2k + 2i - 2] \)
I(i) when \( W = [2k - 2i - 1, 2k - 2i] \cup [2k - 2i + 3, 2k + 2i] \)
I(ii) when \( W = [2k - 2i, 2k + 2i - 3] \)
I(ii) when \( W = [2k - 2i, 2k - 2i + 1] \cup [2k - 2i + 4, 2k + 2i - 1] \) and \( i > 1 \).

Step 1. (Complete exploitation of inductive hypothesis \((11, m)\).)

Consider \( B^m_Z - C^m_W \). In every case it can be observed that \( c(W) \subseteq c(Z) \) whence \( c(W) \subseteq Z \cap W \). By inductive hypothesis \((11, m)\) we have \( B^m_{c(W)} \geq C^m_{c(W)} \), thus

\[
B^m_Z - C^m_W = B^m_Z - B^m_{c(W)} + B^m_{c(W)} - C^m_W \\
\geq B^m_Z - B^m_{c(W)} + C^m_{c(W)} - C^m_W \\
= B^m_{Z_1} - C^m_{W_1},
\]

where \( Z_1 = Z \setminus c(W) \) and \( W_1 = W \setminus c(W) \).

To prove the results it now suffices to show \( B^m_{Z_1} \geq C^m_{W_1} \). We remark that \( Z = Z_1 \) and \( W = W_1 \) if \( 2k \notin W \), and that in the vast majority of cases \( Z_1 \) is the union of two disjoint intervals and \( s(Z_1) \) is a two element set while \( W_1 \) is a single interval with \( s(W_1) \) a one element set. (In any case, in all cases \( s(W_1) \) is a one element set, and \( Z_1 \) and \( W_1 \) are either single intervals or unions of two disjoint intervals.) Further, unless \( 2k \notin W \), in each case \( t(Z_1) = t(W_1) \).

Step 2. (To be used only in the case where \( t(Z_1) = t(W_1) \) otherwise, set \( Z_2 = Z_1 \)
and \( W_2 = W_1 \) and go to Step 3. Step 2 is a simple use of whichever of inductive
hypothesis \((5', m)\) or Claims \((1, 2, 3 \text{ or } 4, m)\) are appropriate to eliminate the
"centre most" element of \( W_1 \).) Consider \( B^m_{Z_1} - C^m_{W_1} \), and let \( j \) be the single element
in \( s(W_1) \) \( \subseteq t(Z_1) \). Recall that \( j \neq 2k \).

(i) If \( j \in [2k+1, 3k] \), observe that also \( j \in Z_1 \), and that \( B^m_{\{j\}} \geq C^m_{\{j\}} \) by inductive
hypothesis \((5', m)\)\( (a), (f) \) and \( (g) \). Now

\[
B^m_{Z_1} - C^m_{W_1} = B^m_{Z_1} - B^m_{\{j\}} + B^m_{\{j\}} - C^m_{W_1} \\
\geq B^m_{Z_1} - B^m_{\{j\}} + C^m_{\{j\}} - C^m_{W_1} \\
= B^m_{Z_2} - C^m_{W_2},
\]

where \( Z_2 = Z_1 \setminus \{j\} \) and \( W_2 = W_1 \setminus \{j\} \).

(ii) If \( j \in [k, 2k - 1] \), observe that \( n - j \in Z_1 \), and that \( B^m_{\{n-j\}} \geq C^m_{\{j\}} \) by Claim
\((3, m+1) \) (if \( j \in O[k, 2k - 1] \)) or Claim \((4, m+1) \) (if \( j \in E[k, 2k - 2] \)). Now

\[
B^m_{Z_1} - C^m_{W_1} \geq B^m_{Z_2} - C^m_{W_2},
\]
where \( Z_2 = Z_1 \setminus \{n-j\} \) and \( W_2 = W_1 \setminus \{j\} \).

(iii) If \( j \in O[1,k-1] \cup E[3k+1,n-2] \), observe that \( j \in Z_1 \), and that \( B_{(j)}^m \geq C_{(j)}^m \) by inductive hypothesis (5'), (m)(a), (b) and (c). Now
\[
B_{Z_2}^m - C_{W_1}^m \geq B_{Z_2}^m - C_{W_2}^m,
\]
where \( Z_2 = Z_1 \setminus \{j\} \) and \( W_2 = W_1 \setminus \{j\} \).

(iv) If \( j \in E[2,k-1] \cup O[3k+1,n-1] \), observe that \( n-j \in Z_1 \), and that \( B_{(n-j)}^m \geq C_{(j)}^m \) by Claim (2, m+1) (if \( j \in E[2,k-1] \)) or Claim (1, m+1) (if \( j \in O[3k+1,n-1] \)). Now
\[
B_{Z_2}^m - C_{W_1}^m \geq B_{Z_2}^m - C_{W_2}^m,
\]
where \( Z_2 = Z_1 \setminus \{n-j\} \) and \( W_2 = W_1 \setminus \{j\} \).

The problem now reduces to demonstrating \( B_{Z_2}^m \geq C_{W_2}^m \). Furthermore, in each case Algorithm 6.8 claims to deal with, the following properties of \( Z_2 \) and \( W_2 \) may be verified

(i) \( W_2 = \emptyset \) and we are finished.

Otherwise we have

(ii) \( t(Z_2) \neq t(W_2) \) and in fact if \( j \in t(Z_2) \cap [1,2k] \) and \( j' \in t(W_2) \cap [1,2k] \), \( j' < j \), (that is, the element of \( Z_2 \) closest to \( 2k \) is closer to \( 2k \) than the element of \( W_2 \) closest to \( 2k \)), and

(iii) \( Z_2 \) is either a single interval or a union of two disjoint intervals and in the latter case the element closest to \( 2k \) in one of these intervals is one closer to \( 2k \) than the element of the other interval which is closest to \( 2k \) and

(iv) \( W_2 \) is either a single interval or a union of two disjoint intervals and in each case the element of \( W_2 \) furthest from \( 2k \) is further from \( 2k \) than the element of \( Z_2 \) furthest from \( 2k \), or there is one less element in \( W_2 \) than in \( Z_2 \) and the element of \( W_2 \) furthest from \( 2k \) is at least as far from \( 2k \) as the element of \( Z_2 \) furthest from \( 2k \). (It is these latter "distance from \( 2k \)" properties which do not hold in the cases not handled by Algorithm 6.8.)

In each case it is a simple matter to verify that full exploitation of Corollaries 5.7 and (48, m) as described in the rest of Algorithm 6.8 completes the proof of the lemma. The remaining step(s) only apply if \( W_2 \neq \emptyset \).

**Step n+1.** (For \( 2 \leq n < \bar{n} \), where \( \bar{n} \) is determined as the smallest integer for which \( W_{\bar{n}} = \emptyset \)).

Let \( j' = \max s(W_n) \) and \( j = \max s(Z_n) \) (notice that \( s(W_2) = \{j'\} \) and \( s(Z_2) = \{j\} \)).

(i) If \( j' < 2k \) and \( j \leq 2k \), \( j' + 1 \leq j' \leq 2k \). Then if \( I = \{i \in [0,j'-1] : j' - i \in W_n \) and \( j' - i \in Z_n\}, 0 \in I, \) and for each \( i \in I, 1 \leq j' - i < j' - i + 1 \leq j - i \leq 2k \) and
\[
B_{(j-i)}^m \geq B_{(j'-i+1)}^m \quad \text{(by Corollary 5.7)}
\]
\[
\geq C_{(j'-i)}^m \quad \text{(by Corollary (6.5, m))}.
\]
So if we set \( Z_{n+1} = Z_n \setminus \{j - i : i \in I\} \) and \( W_{n+1} = W_n \setminus \{j' - i : i \in I\} \),
then
\[
B_{Z_n}^m - C_{W_n}^m \geq B_{Z_{n+1}}^m - C_{W_{n+1}}^m.
\]
(ii) If \(j > 2k\) and \(j \geq 2k, 2k \leq j \leq j'-1\). Then if \(I = \{i \in [0, n-j'] : j+i \in W_n\}\), \(0 \in I\), and for each \(i \in I\), 
\[
B_{j+i}^m \geq B_{j'+i-1}^m \quad \text{(by Corollary 5.7)} \\
\geq C_{j+i}^m \quad \text{(by Corollary (6.5, m)).}
\]
So if we set \(Z_{n+1} = Z_n \setminus \{j+i : i \in I\}\) and \(W_{n+1} = W_n \setminus \{j'+i : i \in I\}\) then
\[
B_{Z_n}^m - C_{W_n}^m \geq B_{Z_{n+1}}^m - C_{W_{n+1}}^m.
\]

(iii) If \(j < 2k\) and \(n-j' > j\) so \(2k < j \leq n-j'-1\). Then if \(I = \{i \in [0, j'-1] : j'-i \in W_n\}\) and \(0 \in I\), and for each \(i \in I\), 
\[
B_{j+i}^m \geq B_{n-j'+i-1}^m \quad \text{(by Corollary 5.7)} \\
\geq C_{j'-i}^m \quad \text{(by Corollary (6.5, m)).}
\]
So if we set \(Z_{n+1} = Z_n \setminus \{j+i : i \in I\}\) and \(W_{n+1} = W_n \setminus \{j'-i : i \in I\}\) then
\[
B_{Z_n}^m - C_{W_n}^m \geq B_{Z_{n+1}}^m - C_{W_{n+1}}^m.
\]

(iv) If \(j > 2k\) and \(j < 2k, j > n-j' \) so \(n-j' + 1 \leq j < 2k\). Then if \(I = \{i \in [0, n-j'-1] : j'+i \in W_n\}\) and \(0 \in I\), and for each \(i \in I\), 
\[
B_{j+i}^m \geq B_{n-j'+i+1}^m \quad \text{(by Corollary 5.7)} \\
\geq C_{j+i}^m \quad \text{(by Corollary (6.5, m)).}
\]
So if we set \(Z_{n+1} = Z_n \setminus \{j-i : i \in I\}\) and \(W_{n+1} = W_n \setminus \{j'+i : i \in I\}\) then
\[
B_{Z_n}^m - C_{W_n}^m \geq B_{Z_{n+1}}^m - C_{W_{n+1}}^m.
\]

We illustrate the use of Algorithm 6.8 in a couple of the more unusual cases (that is, those far removed from the very straightforward and typical instances when \(Z_1\) is the union of two disjoint intervals with \(s(Z_1)\) a two element set, and \(W_1\) is a single interval).

**Example 6.9.** In the proof of I(vii) we consider the case where \(W = [1, n-1] \cap ([2i-2, 2i-1] \cup [2i+2, n-2i+1])\) and \(Z = X_{2k-2i+1} = [2i, n-2i+1],\) for \(i \in [1, (k+1)/2].\)

After Step 1 we found that \(Z_1 = Z \setminus [2i+2, n-2i-2] = [2i, 2i+1] \cup [n-2i-1, n-2i+1]\) and \(W_1 = W \setminus [2i+2, n-2i-2] = [1, n-1] \cap ([2i-2, 2i-1] \cup [n-2i-1, n-2i+1]).\)

For the application of Step 2,
\[
j = n-2i-1 \in \ \left\{ \begin{array}{ll}
O[2k+1, 3k], & \text{if } i \in [(k-1)/2, (k+1)/2] \\
O[3k+1, n-1], & \text{if } i \in [1, (k-2)/2].
\end{array} \right.
\]

Thus \(W_2 = [1, n-1] \cap ([2i-2, 2i-1] \cup [n-2i, n-2i+1])\) and
\[
Z_2 = \ \left\{ \begin{array}{ll}
[2i, 2i+1] \cup [n-2i, n-2i+1], & \text{if } i \in [(k-1)/2, (k+1)/2] \\
\{2i\} \cup [n-2i-1, n-2i+1], & \text{if } i \in [1, (k-1)/2].
\end{array} \right.
\]
For the application of Step 3, \( j' = \max \ s(W_2) = n - 2i > 2k \) and

\[
j = \max \ s(Z_2) = \begin{cases} 
2i + 1 < 2k, & \text{if } i \in [(k-1)/2, (k+1)/2] \\
n - 2i - 1 \geq 2k, & \text{if } i \in [1, (k-2)/2].
\end{cases}
\]

It follows that \( I = [0,1] \) in each case, \( W_3 = W_2 \setminus [n - 2i, n - 2i + 1] = [1, n - 1] \cap [2i - 2, 2i - 1] \) in each case, and

\[
Z_3 = \begin{cases} 
Z_2 \setminus [2i, 2i + 1] = [n - 2i, n - 2i + 1], & \text{if } i \in [(k-1)/2, (k+1)/2] \\
Z_2 \setminus [n - 2i - 1, n - 2i] = [2i, n - 2i + 1], & \text{if } i \in [1, (k-2)/2].
\end{cases}
\]

For the application of Step 4, \( j' = \max \ s(W_3) = 2i - 1 < 2k \) and

\[
j = \max \ s(Z_3) = \begin{cases} 
n - 2i > 2k, & \text{if } i \in [(k-1)/2, (k+1)/2] \\
2i \leq 2k, & \text{if } i \in [1, (k-2)/2].
\end{cases}
\]

Now

\[
I = \begin{cases} 
[0,1], & \text{if } i \in [(k-1)/2, (k+1)/2] \\
\{0\}, & \text{if } i \in [1, (k-2)/2],
\end{cases}
\]

so

\[
W_4 = \begin{cases} 
W_3 \setminus [2i - 2, 2i - 1] = \emptyset, & \text{if } i \in [(k-1)/2, (k+1)/2] \\
W_3 \setminus \{2i - 1\} = \begin{cases} 
\{2i - 2\}, & \text{if } i \in [2, (k-2)/2] \\
\emptyset, & \text{if } i = 1
\end{cases}
\end{cases}
\]

and

\[
Z_4 = \begin{cases} 
Z_3 \setminus [n - 2i, n - 2i + 1] = \emptyset, & \text{if } i \in [(k-1)/2, (k+1)/2] \\
Z_3 \setminus \{2i\} = \{n - 2i + 1\}, & \text{if } i \in [1, (k-2)/2].
\end{cases}
\]

We are now finished except in the case where \( i \in [2, (k-2)/2] \) when \( W_4 = \{2i - 2\} \) and \( Z_4 = \{n - 2i + 1\} \). Here, for the application of Step 5, \( j' = 2i - 2, j = n - 2i + 1, I = \{0\}, W_5 = Z_5 = \emptyset \) and the algorithm completes the task.

**Example 6.10.** In the proof of I(v) we consider the case where

\[
W = [4i - 1, n - 1] \quad \text{and} \quad Z = X_{2k+2i-1} = [2i - 1, n - 2i],
\]

for \( i \in [1, (k+1)/2] \).

After Step 1 we found that \( Z_1 = Z \setminus [4i - 1, n - 4i + 1] = [2i - 1, 4i - 2] \cup [n - 4i + 2, n - 2i] \) and \( W_1 = W \setminus [4i - 1, n - 4i + 1] = [n - 4i + 2, n - 1] \).

This example will illustrate the “typical” behaviour if \( i \in [2, k/2] \) (and is included because of this). For the case \( i = 1 \), however, we have

\[
Z_1 = \{1, 2\} \cup \{n - 2\} \quad \text{and} \quad W_1 = \{n - 2, n - 1\},
\]

which, since \( 1 \in Z_1 \) and \( n - 1 \in W_1 \), would normally be worrying. (Except that \( W_1 \) has one less element than \( Z_1 \).)

For the application of Step 2, \( j = n - 4i + 2 \in \{E[2k + 2, 3k], \quad \text{for } i \in [(k + 2)/4, (k + 1)/2] \\
E[3k + 1, n - 2], \quad \text{for } i \in [1, (k + 1)/4].\} \)
Thus, in either case, \( W_2 = [n - 4i + 3, n - 1] \) and
\[
Z_2 = \begin{cases} 
[2i - 1, 4i - 2] \cup [n - 4i + 3, n - 2i], & \text{for } i \in [2, (k + 1)/2] \\
\{1, 2\}, & \text{for } i = 1.
\end{cases}
\]

For the application of Step 3, \( j' = \max \ s(W_2) = n - 4i + 3 > 2k \) and \( j = \max \ s(Z_2) = 4i - 2 \) in both cases. But \( 4i - 2 = 2k \) and \( n - 4i + 3 = 2k + 1 \) if \( i = (k + 1)/2 \), and \( 4i - 2 < 2k \) if \( i < (k + 1)/2 \). It follows that
\[
I = \begin{cases} 
[0, k - 1], & \text{if } i = (k + 1)/2 \\
[0, 2i - 1], & \text{if } i \in [2, k/2] \\
\{0\}, & \text{if } i = 1.
\end{cases}
\]

Thus
\[
W_3 = \begin{cases} 
[n - 2i + 2, n - 1] = [3k + 1, n - 1], & \text{if } i = (k + 1)/2 \\
[n - 2i + 3, n - 1], & \text{if } i \in [2, k/2] \\
\emptyset, & \text{if } i = 1.
\end{cases}
\]

and
\[
Z_3 = \begin{cases} 
[2i - 1, 4i - 3] = [k, 2k - 1], & \text{if } i = (k + 1)/2 \\
[n - 4i + 3, n - 2i], & \text{if } i \in [2, k/2] \\
\{1\}, & \text{if } i = 1.
\end{cases}
\]

(So the case \( i = 1 \) is finished.)

For the application of Step 4,
\[
j' = \max \ s(W_3) = \begin{cases} 
3k + 1 > 2k, & \text{if } i = (k + 1)/2 \\
n - 2i + 3 > 2k, & \text{if } i \in [2, k/2]
\end{cases}
\]

and
\[
j = \max \ s(Z_3) = \begin{cases} 
2k - 1 < 2k, & \text{if } i = (k + 1)/2 \\
n - 4i + 3 \geq 2k, & \text{if } i \in [2, k/2].
\end{cases}
\]

It follows that
\[
I = \begin{cases} 
[0, k - 2], & \text{if } i = (k + 1)/2 \\
[0, 2i - 4], & \text{if } i \in [2, k/2].
\end{cases}
\]

In all cases now \( W_4 = \emptyset \) and we are finished.

**Algorithm 6.11.** This produces a proof in those cases excepted in Algorithm 6.8.

**Step 1.** This step is identical to Step 1 of Algorithm 6.8; that is, exploit inductive hypothesis (11, m) to its fullest extent to reduce the problem of showing \( B_{Z_1}^m \geq C_{W_1}^m \) to showing \( B_{Z_1}^m \geq C_{W_1}^m \), where \( Z_1 = Z \setminus c(W) \) and \( W_1 = W \setminus c(W) \).

**Step 2.** This step, in effect, exploits inductive hypothesis (5', m)(a), (f) and (g) applied to \( j \in [2k + 1, 3k] \) to its fullest extent. Formally, consider \( B_{Z_1}^m - C_{W_1}^m \). If we set \( P_1 = [2k + 1, 3k] \cap Z_1 \cap W_1 \subseteq Z_1 \), \( P_1 \subseteq W_1 \), and, by inductive hypothesis (5', m)(a), (f) and (g) we have \( B_{P_1}^m \geq C_{P_1}^m \), thus
\[
B_{Z_1}^m - C_{W_1}^m = B_{Z_1}^m - B_{P_1}^m + B_{P_1}^m - C_{W_1}^m \\
\geq B_{Z_1}^m - B_{P_1}^m + C_{P_1}^m - C_{W_1}^m \\
= B_{Z_2}^m - C_{W_2}^m,
\]
where \( Z_2 = Z_1 \setminus P_1 \) and \( W_2 = W_1 \setminus P_1 \). To prove the result it now suffices to show \( B_{Z_2}^m \geq C_{W_2}^m \).

**Step 3.** This step, in effect, exploits Claims (3, m+1) and (4, m+1) to their fullest extent. Formally, consider \( B_{Z_2}^m - C_{W_2}^m \). Let \( J_2 = \{ j \in [1, k] : 2k + j \in Z_2 \text{ and } 2k - j \in W_2 \} \), and let \( P_2 = \{ 2k + j : j \in J_2 \} \) and \( Q_2 = \{ 2k - j : j \in J_2 \} \). Note that \( P_2 \subseteq Z_2 \) and \( Q_2 \subseteq W_2 \). Note also that if \( 2k + j \in P_2 \) and \( j \) is odd, \( 2k - j \in Q_2 \). Thus, by Claim (3, m+1), \( B_{Z_2}^m \geq C_{W_2}^m \). Similarly if \( 2k + j \in P_2 \) and \( j \) is even, \( 2k - j \in Q_2 \) and \( B_{Z_2}^m \geq C_{Q_2}^m \) by Claim (4, m+1). It follows that \( B_{P_2}^m \geq C_{Q_2}^m \).

Now
\[
B_{Z_2}^m - C_{W_2}^m = B_{Z_2}^m - B_{P_2}^m + B_{P_2}^m - C_{W_2}^m
\geq B_{Z_2}^m - B_{P_2}^m + C_{Q_2}^m - C_{W_2}^m
= B_{Z_2}^m - C_{W_2}^m,
\]

where \( Z_3 = Z_2 \setminus P_2 \) and \( W_3 = W_2 \setminus Q_2 \). To prove the result it now suffices to show \( B_{Z_3}^m \geq C_{W_3}^m \).

It transpires that in all those cases excepted in Algorithm 6.8, application of Algorithm 6.11 to Step 3 either yields \( Z_3 = W_3 = \emptyset \) and we are finished, or \( Z_3 = \{ j \} \) with \( j \leq 2k \) and \( W_3 = \{ j - 2 \} \). In the latter case
\[
C_{W_3}^m = C_{(j-2)}^m \leq B_{(j-1)}^m \quad \text{(by Corollary (6.5, m))}
\leq B_{\{ j \}}^m \quad \text{(by Corollary 5.7)}
= B_{Z_3}^m,
\]

and we are finished. Again we will illustrate with one of the cases.

**Example 6.12.** Consider the expected case in \(\Pi^i(i)\) where \( W = [2k - 2i - 1, 2k - 2i] \cup [2k - 2i + 3, 2k + 2i] \) and \( Z = X_{2i} = [2k - 2i + 2, 2k + 2i + 1] \) for \( i \in [1, (k-1)/2] \).

After Step 1 we found that
\[
Z_1 = \begin{cases} 
Z \setminus [2k - 2i + 3, 2k + 2i - 3] \\
= [2k - 2i + 2] \cup [2k + 2i - 2, 2k + 2i + 1], & \text{for } i > 1 \\
= [2k, 2k + 3], & \text{for } i = 1.
\end{cases}
\]

and
\[
W_1 = \begin{cases} 
W \setminus [2k - 2i + 3, 2k + 2i - 3] \\
= [2k - 2i - 1, 2k - 2i] \cup [2k + 2i - 2, 2k + 2i], & \text{for } i > 1 \\
= [2k - 3, 2k - 2, 2k + 1, 2k + 2], & \text{for } i = 1.
\end{cases}
\]

After Step 2 we find
\[
Z_2 = \begin{cases} 
Z_1 \setminus [2k + 2i - 2, 2k + 2i] = \{ 2k - 2i + 2, 2k + 2i + 1 \}, & \text{for } i > 1 \\
= [2k, 2k + 3], & \text{for } i = 1.
\end{cases}
\]

and
\[
W_2 = \begin{cases} 
W_1 \setminus [2k + 2i - 2, 2k + 2i] = \{ 2k - 2i - 1, 2k - 2i \}, & \text{for } i > 1 \\
= [2k + 1, 2k + 2], & \text{for } i = 1.
\end{cases}
\]

After Step 3 we find \( Z_3 = Z_2 \setminus \{ 2k + 2i + 1 \} = \{ 2k - 2i + 2 \} \) for \( i \in [1, (k-1)/2] \)
and \( W_3 = W_2 \setminus \{ 2k - 2i - 1 \} = \{ 2k - 2i \} \) for \( i \in [1, (k-1)/2] \), as required.

This completes the proof of Lemma (6.7, m).
Proof of Claim $(5', m+1)(b)$, $(6', c), (d), (e), (f), (g), (h)$ and $(i)$. 

(b) We wish to show that \( \left| (B^m_{m+1})^{(2i-1)} \right| \geq \left| (C^m_{m+1})^{(2i-1)} \right| \) for \( i \in [1, k/2] \), where \( (a, b)^{2i-1} \neq (x, y)^{2i-1} \); that is, we wish to show \( B^m_{X_{2i-1}} \geq C^m_{Y_{2i-1}} \) for \( i \in [1, k/2] \), where \( Y_{2i-1} \neq X_{2i-1} \). Let \( i \in [1, k/2] \). By Corollary 5.13 we can choose \( W_{2i-1} \in T_{2i-1} \) such that

\[ C^m_{W_{2i-1}} \geq C^m_{Y_{2i-1}}. \]

Further, if \( W_{2i-1} = X_{2i-1} \), \( B^m_{X_{2i-1}} \geq C^m_{X_{2i-1}} = C^m_{W_{2i-1}} \) by Claim $(5', m+1)(a)$, and if \( W_{2i-1} \neq X_{2i-1} \), \( W_{2i-1} \in T_{2i-1} \setminus \{ X_{2i-1} \} \) and so

\[ B^m_{X_{2i-1}} \geq C^m_{W_{2i-1}} \]

by Lemma (6.7, m)(I)(i).

Hence in all cases, \( B^m_{X_{2i-1}} \geq C^m_{W_{2i-1}} \geq C^m_{Y_{2i-1}} \) as required.

(c) Is proved similarly to (b), using Corollary 5.13, Claim $(5', m+1)(a)$ and Lemma (6.7, m)(I)(ii).

(d) We wish to show that \( \left| (B^m_{m+1})^{(2i)} \right| \geq \left| (C^m_{m+1})^{(2i)} \right| \) for \( i \in [1, (k-1)/2] \), where \( (a, b)^{2i} \neq (x, y)^{2i} \) and \( (a, b)^{2i} \neq (2k-2i, 2k+2i-1) \); that is, we wish to show \( B^m_{X_{2i}} \geq C^m_{Y_{2i}} \) for \( i \in [1, (k-1)/2] \), where \( Y_{2i} \neq X_{2i} \) and \( Y_{2i} \neq [2k-2i, 2k+2i-1] \). Let \( i \in [1, (k-1)/2] \). Since \( Y_{2i} \neq [2k-2i, 2k+2i-1] \), by Corollary 5.13 we can choose \( W_{2i} \in T_{2i} \) such that

\[ C^m_{W_{2i}} \geq C^m_{Y_{2i}}. \]

Further, if \( W_{2i} = X_{2i} \), \( B^m_{X_{2i}} \geq C^m_{X_{2i}} = C^m_{W_{2i}} \) by Claim $(5', m+1)(a)$, and if \( W_{2i} \neq X_{2i} \), \( W_{2i} \in T_{2i} \setminus \{ X_{2i} \} \) and so

\[ B^m_{X_{2i}} \geq C^m_{W_{2i}}, \] by Lemma (6.7, m)(I)(iii).

Hence in all cases, \( B^m_{X_{2i}} \geq C^m_{W_{2i}} \geq C^m_{Y_{2i}} \) as required.

(e) Is proved similarly to (d), using Corollary 5.13, Claim $(5', m+1)(a)$ and Lemma (6.7, m)(I)(iv).

(f) Is proved similarly to (b), using Corollary 5.13, Claim $(5', m+1)(a)$ and Lemma (6.7, m)(I)(v).

(g) Is proved similarly to (b), using Corollary 5.13, Claim $(5', m+1)(a)$ and Lemma (6.7, m)(I)(vi).

(h) We wish to show that \( \left| (B^m_{m+1})^{(2k-2i+1)} \right| \geq \left| (C^m_{m+1})^{(2k-2i+1)} \right| \) for \( i \in [1, (k+1)/2] \), where \( (a, b)^{2k-2i+1} \neq (x, y)^{2k-2i+1} \); that is, we wish to show \( B^m_{X_{2k-2i+1}} \geq C^m_{Y_{2k-2i+1}} \) for \( i \in [1, (k+1)/2] \), where \( Y_{2k-2i+1} \neq X_{2k-2i+1} \). Let \( i \in [1, (k+1)/2] \). Since \( Y_{2k-2i+1} \neq X_{2k-2i+1} \), by Corollary 5.13 we can choose \( W_{2k-2i+1} \in T'_{2k-2i+1} \) such that

\[ C^m_{W_{2k-2i+1}} \geq C^m_{Y_{2k-2i+1}}. \]

Thus

\[ B^m_{X_{2k-2i+1}} \geq C^m_{W_{2k-2i+1}} \geq C^m_{Y_{2k-2i+1}} \]

by Lemma (6.7, m)(I)(vii).

(i) Is proved similarly to (h), using Corollary 5.13 and Lemma (6.7, m)(I)(viii).
Proof of Claim (6, m+1). For j \in [k, 2k - 1] we have \langle a, b \rangle^j = \langle x, y \rangle^j and (a, b)^{n-j} = \langle x, y \rangle^{n-j} by assumption.

(a) Let i \in [1, (k + 1)/2]. Then

\[
\left| (B^{m+1})^{(2k-2i+1)} \right| + \left| (B^{m+1})^{(2k+2i-1)} \right| - \left| (C^{m+1})^{(2k-2i+1)} \right| - \left| (C^{m+1})^{(2k+2i-1)} \right|
\]
\[
= (B^m - C^m)[2i, n - 2i + 1] + (B^m - C^m)[2i - 1, n - 2i]
\]
\[
= (B^m - C^m)[2i - 1, n - 2i + 1] + (B^m - C^m)[2i, n - 2i]
\]
\[
\geq 0,
\]
by inductive hypothesis (11, m).

(b) Now let i \in [1, k/2]. Then

\[
\left| (B^{m+1})^{(2k-2i)} \right| + \left| (B^{m+1})^{(2k+2i)} \right| - \left| (C^{m+1})^{(2k-2i)} \right| - \left| (C^{m+1})^{(2k+2i)} \right|
\]
\[
= (B^m - C^m)[2i, n - 2i - 1] + (B^m - C^m)[2i + 1, n - 2i]
\]
\[
= (B^m - C^m)[2i, n - 2i] + (B^m - C^m)[2i + 1, n - 2i - 1]
\]
\[
\geq 0, \quad \text{by inductive hypothesis (11, m).}
\]

Proof of Claim (7, m+1). Firstly note that under the given restrictions, in each case we wish to show

\[
\left| (B^{m+1})^{(j)} \right| + \left| (B^{m+1})^{(s)} \right| - \left| (C^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(s)} \right| \geq 0.
\]

Now in each case

\[
\left| (B^{m+1})^{(j)} \right| + \left| (B^{m+1})^{(s)} \right| - \left| (C^{m+1})^{(j)} \right| - \left| (C^{m+1})^{(s)} \right|
\]
\[
= B^m_{X_j} + B^m_{X_s} - C^m_{Y_j} - C^m_{Y_s} \quad \text{(where } Y_s \neq X_s \text{ and } Y_j \text{ is as specified)}
\]
\[
\geq B^m_{X_j} + B^m_{X_s} - C^m_{Y_j} - C^m_{W},
\]
for some W \in T'_s by (Corollary 5.13).

Thus it will suffice to show

\[
(6.1) \quad B^m_{X_j} - C^m_{Y_j} + B^m_{X_s} - C^m_{W} \geq 0
\]

for all W \in T'_s, where Y_j is as specified. We call a proof of (6.1) for any given W \in T'_s a **direct proof** (of a specific subcase of Claim (7, m+1)).

Note further that for j = 2i, i \in [1, (k - 1)/2], Y_j = [2k - 2i, 2k + 2i - 1] (as is the case throughout Claim (7, m+1) (a), (b) and (c)) and W = \in T'_s

\[
B^m_{X_j} - C^m_{Y_j} + B^m_{X_s} - C^m_{W}
\]
\[
= B^m_{X_2} - C^m_{[2k-2i,2k+2i-1]} + B^m_{X_s} - C^m_{W}
\]
\[
= B^m_{X_2} - B^m_{X_{n-2i}} + B^m_{X_{n-2i}} - C^m_{[2k-2i,2k+2i-1]} + B^m_{X_s} - C^m_{W}
\]
\[
\geq B^m_{X_2} - B^m_{X_{n-2i}} + B^m_{X_s} - C^m_{W} \quad \text{(by Lemma (6.7, m)(1)(ii))}
\]
\[
= B^m_{[2k-2i+2,2k+2i+1]} - B^m_{[2k-2i+1,2k+2i]} + B^m_{X_s} - C^m_{W}
\]
\[
= B^m_{[2k+2i+1]} - B^m_{[2k-2i+1]} + B^m_{X_s} - C^m_{W}.
\]
Thus if we set $T'_s = (T'_s)_1 \cup (T'_s)_2$ for some choices of $(T'_s)_1$ and $(T'_s)_2$ (depending on $s$), and we can show (6.1) to be true for all $W \in (T'_s)_1$ and

\begin{align*}
(6.2) \quad B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{X_s} - C^m_W \geq 0
\end{align*}

for all $W \in (T'_s)_2$, (a proof of which, for some $W \in (T'_s)_2$, will be called an indirect proof of a specific subcase of Claim (7, $m+1$)(a), (b) or (c)), we will have completed the proof of Claim (7, $m+1$)(a), (b) and (c).

Similarly, throughout Claim (7, $m+1$) (d), (e) and (f) we have $j = n - 2i + 1$, $i \in [1, k/2]$ and $Y_j = [2k - 2i + 1, 2k + 2i - 2]$. So if $W \in (T'_s)_2$

\begin{align*}
B^m_{X_j} - C^m_{Y_j} + B^m_{X_s} - C^m_W
&= B^m_{X_{n-2i+1}} - C^m_{[2k-2i+1, 2k+2i-2]} + B^m_{X_s} - C^m_W \\
&= B^m_{X_{n-2i+1}} - B^m_{X_{2i-1}} + \left(B^m_{X_{2i-1}} - C^m_{[2k-2i+1, 2k+2i-2]}\right) + B^m_{X_s} - C^m_W \\
&\geq B^m_{X_{n-2i+1}} - B^m_{X_{2i-1}} + B^m_{X_s} - C^m_W \quad \text{(by Lemma (6.7, m)(1)(i))} \\
&= B^m_{[2k-2i+3, 2k+2i]} - B^m_{[2k-2i+2, 2k+2i-1]} + B^m_{X_s} - C^m_W \\
&= B^m_{\{2k+2i\}} - B^m_{\{2k-2i+2\}} + B^m_{X_s} - C^m_W.
\end{align*}

Thus again, if we can show (6.1) to be true for all $W \in (T'_s)_1$, and

\begin{align*}
(6.3) \quad B^m_{\{2k+2i\}} - B^m_{\{2k-2i+2\}} + B^m_{X_s} - C^m_W \geq 0
\end{align*}

for all $W \in (T'_s)_2$ (an indirect proof of the specific subcases of Claim (7, $m+1$)(d), (e) and (f)) we will have completed the proof of Claim (7, $m+1$)(d), (e) and (f).

Firstly we tackle the (easy) indirect cases.

(i) **Proof of Claim (7, $m+1$)(a) indirect cases.** Here $s = n - 2i$ and $i \in [1, (k - 1)/2]$ so $X_s = X_{n-2i}$. We set $(T'_{n-2i})_1 = \emptyset$ and $(T'_{n-2i})_2 = T'_{n-2i}$. (So there are no direct cases to prove here). Let $W \in (T'_{n-2i})_2 = T'_{n-2i}$. Then

\begin{align*}
B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{X_s} - C^m_W \\
&= B^m_{X_{2i}} - B^m_{X_{n-2i}} + B^m_{X_s} - C^m_W \\
&= B^m_{X_{2i}} - C^m_W \quad \text{(since $X_s = X_{n-2i}$)} \\
&\geq 0 \quad \text{(by Lemma (6.7, m)(II)(i)).}
\end{align*}

(ii) **Proof of Claim (7, $m+1$)(d) indirect cases.** Here $s = 2i - 1$ and $i \in [1, k/2]$ so $X_s = X_{2i-1}$. We set $(T'_{2i-1})_1 = \emptyset$ and $(T'_{2i-1})_2 = T'_{2i-1}$ (So there are no direct cases to prove here). Let $W = (T'_{2i-1})_2 = T'_{2i-1}$. Then

\begin{align*}
B^m_{\{2k+2i\}} - B^m_{\{2k-2i+2\}} + B^m_{X_s} - C^m_W \\
&= B^m_{X_{n-2i+1}} - B^m_{X_{2i-1}} + B^m_{X_s} - C^m_W \\
&= B^m_{X_{n-2i+1}} - C^m_W \quad \text{(since $X_s = X_{2i-1}$)} \\
&\geq 0 \quad \text{(by Lemma (6.7, m)(II)(ii)).}
\end{align*}
(iii) Proof of Claim (7, m+1)(c) indirect cases. Here \( s = 2k + 2i \) and \( i \in [1, (k - 1)/2] \) so \( X_s = X_{2k+2i} = [2i + 1, n - 2i] \). We set \((T'_{2k+2i})_1 = \{ [2i - 1, n - 2i - 2], [2i - 1, 2i] \cup [2i + 3, n - 2i], [1, n - 1] \cap (\{2i+1, n-2i-2\} \cup [n-2i+1, n-2i+2]) \}
\text{ and } (T'_{2k+2i})_2 = T'_{2k+2i} \setminus (T'_{2k+2i})_1 \). Let \( W \in (T'_{2k+2i})_2 \). Then

\[
B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{X_s} - C^m_W
= B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{\{2i+1, n-2i\}} - B^m_{\{2i+2, n-2i+1\}} + B^m_{\{2i+2, n-2i-1\}} - C^m_W
\geq B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{\{2i+1\}} - B^m_{\{n-2i+1\}} \text{ (by Lemma (6.7, m)(II)(iii))}
= B^m_{\{2i+1, n-2i-2\}} - B^m_{\{2i, n-2i+1\}} + B^m_{\{2k-2i, 2k+2i+1\}} - B^m_{\{2k-2i+3, 2k+2i\}} \text{ (note that the formula is still valid in the case } 2i + 1 = k \text{ and } 2k + 2i + 1 = 3k \)
= B^m_{\{2k-2i, 2k-2i+1, 2k-2i+2, 2k+2i+1\}} - B^m_{\{2i, n-2i-1, n-2i, n-2i+1\}} 
\geq 0
\]

This follows by Lemma 6.2 since \( i \in [1, (k - 1)/2] \), \( \{2k - 2i, 2k - 2i + 1, 2k - 2i + 2, 2k + 2i + 1\} \subseteq [k, 3k] \) and \( \{2i, n-2i-1, n-2i, n-2i+1\} \subseteq [1, k] \cup [3k, n-1] \).

(iv) Proof of Claim (7, m+1)(b) indirect cases. Here \( s = 2k - 2i \) and \( i \in [1, (k - 1)/2] \) so \( X_s = X_{2k-2i} = [2i, n - 2i - 1] \). We set \((T'_{2k-2i})_1 = \{ [2i + 2, n - 2i + 1], [2i, n - 2i - 3] \cup [n - 2i, n - 2i + 1] \} \text{ and } (T'_{2k-2i})_2 = T'_{2k-2i} \setminus (T'_{2k-2i})_1 \). Let \( W \in (T'_{2k-2i})_2 \). Then

\[
B^m_{\{2k+2i+1\}} - B^m_{\{2k-2i+1\}} + B^m_{X_s} - C^m_W
= -B^m_{\{2k-2i-1, n-2i, n-2i+1\}} + B^m_{\{2i, n-2i-1\}} - B^m_{\{2i-1, n-2i-2\}} + B^m_{\{2i-1, n-2i-2\}} - C^m_W \text{ (as in case (iii) above)}
\geq B^m_{\{n-2i+1\}} - B^m_{\{2i-1\}} - B^m_{\{2i, n-2i-1, n-2i, n-2i+1\}} \text{ (by Lemma (6.7, m)(II)(iv))}
= \begin{cases} 
B^m_{\{k, 3k-1\}} - B^m_{\{k+3, 3k-2\}} - B^m_{\{k+1, 3k+1, 3k+2\}}, & \text{if } i = (k - 1)/2 \\
B^m_{\{2k-2i+1, 2k+2i+1\}} - B^m_{\{2k-2i+3, 2k+2i+1\}} & \text{as in case (iii) above).}
\end{cases}
\]

(v) Proof of Claim (7, m+1)(e) indirect cases. Here \( s = 2k-2i+1 \) and \( i \in [1, k/2] \) so \( X_s = X_{2k-2i+1} = [2i, n - 2i + 1] \). We set

\[
(T'_{2k-2i+1})_1 = \begin{cases} 
\{ [2i - 2, n - 2i - 1], \\
[2i - 2, 2i - 1] \cup [2i + 2, n - 2i + 1], & \text{for } i > 1 \\
[1, n - 3], \{1\} \cup [4, n-1] = T'_2, & \text{for } i = 1 
\end{cases}
\]
and \((T_{2k-2i+1})_2 = T_{2k-2i+1}' \setminus (T_{2k-2i+1})_1\). Let \(W \in (T_{2k-2i+1})_2\) and note \((T_{2k-2i+1})_2 = \emptyset\) if \(i = 1\), so we may assume \(i \in [2, k/2]\). Thus
\[
B_m^{n \{2k+2i\}} - B_m^{n \{2k-2i+2\}} + B_m^{n \{2i,n-2i+1\}} - B_m^{n \{2i+1,n-2i+1\}} \leq B_m^{n \{2k+3\}} - B_m^{n \{2k-3\}} + B_m^{n \{2i\}} - B_m^{n \{i\}} \quad \text{(by Lemma (6.7, m)(II)(v))}
\]
\[
= \begin{cases} 
B_m^{n \{k+1,3k\}} - B_m^{n \{k-1,3k\}} & \text{if } i = k/2 \\
B_m^{n \{2i+1,n-2i\}} - B_m^{n \{2i-1,n-2i+1\}} + B_m^{n \{2i+2,2k+2i\}} & \text{if } i < k/2 \\
B_m^{n \{k+1,3k+1\}} - B_m^{n \{k-1,3k-1\}} & \text{if } i = k/2 \\
B_m^{n \{2i+2,2k+2i+1\}} - B_m^{n \{2i-2,2i-1,2i,n-2i+1\}} & \text{if } 2i \leq k - 1
\end{cases}
\]
\[
\geq 0 \quad \text{(as in case (iii) above)}.
\]

(vi) Proof of Claim (7, m+1)(f) indirect cases. Here \(s = 2k+2i-1\) and \(i \in [1, k/2]\) so \(X_s = X_{2k+2i-1} = [2i-1, n-2i]\). We set
\[
(T_{2k+2i-1})_1 = \begin{cases} 
\{[2i-1, n-2i-2] \cup [n-2i+1, n-2i+2], \\
[2i+1, n-2i+2], [2i-3, 2i-2] \cup [2i+1, n-2i]\}, & \text{for } i > 1 \\
\{[3, n-1], [1, n-4] \cup \{n-1\}\}, & \text{for } i = 1
\end{cases}
\]
and \((T_{2k+2i-1})_2 = T_{2k+2i-1}' \setminus (T_{2k+2i-1})_1\). Let \(W \in (T_{2k+2i-1})_2\) and note that \((T_{2k+2i-1})_2 = \emptyset\) if \(i = 1\), so we may assume \(i \in [2, k/2]\). Thus
\[
B_m^{n \{2k+2i\}} - B_m^{n \{2k-2i+2\}} + B_m^{n \{2i,n-2i+1\}} - C_m^W
\]
\[
= -B_m^{n \{2i-2,2i-1,2i,n-2i+1\}} + B_m^{n \{2i-1, n-2i\}} - B_m^{n \{2i-2,n-2i-1\}} + B_m^{n \{2i-2,2i-1,2i,n-2i+1\}} - C_m^W \quad \text{(as in case (v) above)}
\]
\[
\geq B_m^{n \{n-2i\}} - B_m^{n \{2i-2\}} + B_m^{n \{2i-2,2i-1,2i,n-2i+1\}} \quad \text{(by Lemma (6.7, m)(II)(vi))}
\]
\[
= B_m^{n \{2k-2i+2,2k+2i\}} - B_m^{n \{2k-2i+1,2k+2i+1\}} - B_m^{n \{2k-2i+1,2k+2i-1\}} - B_m^{n \{2k-2i+1,2k+2i\}} - B_m^{n \{2k-2i+1,2k+2i+1\}} - B_m^{n \{2i-2,2i-1,2i,n-2i+1\}}
\]
\[
\geq 0 \quad \text{(as in case (iii) above)}.
\]

Finally it remains to prove Claim (7, m+1) in the complicated direct cases. As we have seen there is nothing to do for (7, m+1)(a) and (d). For (7, m+1)(b) and (c) we have \(j = 2i, i \in [1, (k-1)/2]\) and \(Y_j = [2k-2i, 2k+2i-1]\). Thus
\[
B_m^{n \{2k+2i\}} - C_m^W + B_m^{n \{2k-2i+2\}} - C_m^W
\]
\[
= B_m^{n \{2k-2i+2,2k+2i+1\}} - C_m^{n \{2k-2i,2k+2i\}} + B_m^{n \{2k-2i,2k+2i\}} - C_m^W
\]
\[
\geq B_m^{n \{2k+2i+1\}} - C_m^{n \{2k+2i+1\}} + B_m^{n \{2k-2i+1\}} - C_m^W
\]
\[
\text{(applying the three steps of Algorithm 6.11 to the first two terms)}
\]
\[
\geq B_m^{n \{2k+2i+1\}} - B_m^{n \{2k+2i\}} + B_m^{n \{2k-2i+1\}} - C_m^W \quad \text{(by Claim (3, m+1))}
\]
\[
= B_m^{n \{2i,2i-1\}} - B_m^{n \{2i,2i-2\}} \quad \text{if } i \in [1, (k-1)/2]\) and \(Y_j = [2k-2i, 2k+2i-1]\). Thus
\[
B_m^{n \{2k-2i+2\}} - C_m^W - 4B_m^{n \{k\}} \quad \text{(as in case (iii) above)}.
\]
It follows that to complete the proofs of (7, m+1)(b) and (c) it will suffice to show

$$B_{X_s}^m - C_W^m \geq 4B_{\{k\}}^{m-1}$$

for all $W \in (T'_3)$. Further for (7, m+1)(e) and (f) we have $j = n-2i+1, \ i \in [1, k/2]$ and $Y_j = [2k - 2i + 1, 2k + 2i - 2]$. Thus

$$B_{X_s}^m - C_{Y_j}^m + B_{X_s}^m - C_W^m$$

$$= B_{[2k-2i+3, 2k+2i]}^m - C_{[2k-2i, 2k+2i-2]}^m + B_{X_s}^m - C_W^m$$

$$\geq B_{[2k+2i]}^m - C_{[2k-2i+2]}^m + B_{X_s}^m - C_W^m$$

(applying the three steps of Algorithm 6.11 to the first two terms)

$$\geq B_{[2k+2i]}^m - B_{[2k+2i-2]}^m + B_{X_s}^m - C_W^m$$ (by Claim (4, m+1))

$$= \begin{cases} 
B_{[2i+1, n-2i]}^m - B_{[2i-1, n-2i+2]}^m + B_{X_s}^m - C_W^m, & \text{if } i > 1 \\
B_{[3, n-2]}^m - B_{[1, n-1]}^m + B_{X_s}^m - C_W^m, & \text{if } i = 1 
\end{cases}$$

$$= \begin{cases} 
B_{X_s}^m - C_W^m - B_{\{i, 1, n-2i+1, n-2i+2\}}, & \text{if } i > 1 \\
B_{X_s}^m - C_W^m - B_{\{1, 2, n-1\}}, & \text{if } i = 1 
\end{cases}$$

$$\geq \begin{cases} 
B_{X_s}^m - C_W^m - 4B_{\{k\}}^{m-1}, & \text{if } i > 1 \text{ (as in (iii) above)} \\
B_{X_s}^m - C_W^m - 3B_{\{k\}}^{m-1}, & \text{if } i = 1 
\end{cases}$$

Thus to complete the proof of (7, m+1) in all remaining cases it suffices to show

$$B_{X_s}^m - C_W^m \geq 4B_{\{k\}}^{m-1}$$

for all $W \in (T'_3)$.

Once more we provide an algorithm to achieve the desired result. The algorithm extends Algorithm 6.11 by also exploiting Claims (1, m+1) and (2, m+1) and the remainder of inductive hypothesis (5, m) to the full as follows:

**Algorithm 6.13.** Steps 1, 2, and 3 are the same as Algorithm 6.11, commencing with $Z = X_s$. So we are now considering $B_{Z_3}^m - C_{W_3}^m$, and wish to show $B_{Z_3}^m - C_{W_3}^m \geq 4B_{\{k\}}^{m-1}$.

**Step 4.** Consider $B_{Z_3}^m - C_{W_3}^m$. If $\{k\} \subseteq Z_3 \cap W_3$ then

$$B_{Z_3}^m - C_{W_3}^m = B_{Z_3}^m - B_{\{k\}}^m + B_{\{k\}}^m - C_{W_3}^m$$

$$\geq B_{Z_3}^m - B_{\{k\}}^m + C_{\{k\}}^m - C_{W_3}^m$$ (by inductive hypothesis (5, m))

$$= B_{Z_3}^m - C_{W_4}^m$$

where $Z_4 = Z_3 \setminus \{k\}$ and $W_4 = W_3 \setminus \{k\}$.

To prove the result it now suffices to show $B_{Z_4}^m - C_{W_4}^m \geq 4B_{\{k\}}^{m-1}$.

**Step 5.** Consider $B_{Z_4}^m - C_{W_4}^m$. If $k \in Z_4$ and $3k \in W_4$ then

$$B_{Z_4}^m - C_{W_4}^m = B_{Z_4}^m - B_{\{k\}}^m + B_{\{3k\}}^m - C_{W_4}^m$$ (by Lemma 6.2)

$$\geq B_{Z_4}^m - B_{\{k\}}^m + C_{\{3k\}}^m - C_{W_4}^m$$ (by inductive hypothesis (5, m))

$$= B_{Z_5}^m - C_{W_5}^m$$
where $Z_5 = Z_4 \setminus \{k\}$ and $W_5 = W_4 \setminus \{3k\}$.

To prove the result it now suffices to show $B_{Z_5}^m - C_{W_5}^m \geq 4B_{\{k\}}^{m-1}$.

The next two steps now change in nature as we no longer concern ourselves about the nature of $Z_6$ or $Z_7$. In all cases to be considered it transpires that $W_5$ is a non-empty subset of $[1, k-1] \cup [3k+1, n-1]$. 

**Step 6.** Consider $B_{Z_5}^m - C_{W_5}^m$. Let $P_5 = W_5 \cap (O[1, k-1] \cup E[3k+1, n-2])$. Now

$$B_{Z_5}^m - C_{W_5}^m = B_{Z_5}^m - B_{P_5}^m + B_{P_5}^m - C_{W_5}^m$$

$$\geq B_{Z_5}^m - B_{P_5}^m + C_{P_5}^m - C_{W_5}^m \quad \text{(by inductive hypothesis (5, m))}$$

$$= B_{Z_6}^m - B_{P_6}^m - C_{W_6}^m$$

where $Z_6 = Z_5 \setminus P_5$, $P_6 = P_5 \setminus Z_5$ and $W_6 = W_5 \setminus P_5$.

To prove the result it now suffices to show $B_{Z_6}^m - B_{P_6}^m - C_{W_6}^m \geq 4B_{\{k\}}^{m-1}$.

**Step 7.** Consider $B_{Z_6}^m - B_{P_6}^m - C_{W_6}^m$, $W_6 \subseteq E[2, k-1] \cup O[3k+1, n-1]$. Let $Q_6 = \{n-j : j \in W_6\}$. Now

$$B_{Z_6}^m - B_{P_6}^m - C_{W_6}^m = B_{Z_6}^m - B_{P_6}^m - B_{Q_6}^m + B_{Q_6}^m - C_{W_6}^m$$

$$\geq B_{Z_6}^m - B_{P_6}^m - B_{Q_6}^m \quad \text{(by Claims (1, m+1) and (2, m+1))}$$

$$= B_{Z_7}^m - B_{P_7}^m - B_{Q_7}^m$$

where $Z_7 = Z_6 \setminus Q_6$, and $Q_7 = Q_6 \setminus Z_6$.

To prove the result it now suffices to show $B_{Z_7}^m - B_{P_7}^m - B_{Q_7}^m \geq 4B_{\{k\}}^{m-1}$.

**Step 8.** Convert each term in $B_{Z_7}^m$, $B_{P_7}$, and $B_{Q_7}^m$, to level $m-1$ using $B_{X_j}^m = B_{X_j}^{m-1}$ and simplify. It will transpire that $B_{Z_7}^m - B_{P_7}^m - B_{Q_7}^m$ will simplify to a sum of terms of the form $B_{\{j\}}^{m-1}$. Furthermore, at least four of the $B_{\{j\}}^{m-1}$ terms to be summed will have $j \in [k, 3k]$, thus the overall result will follow using Lemma 6.2.

We will illustrate the use of Algorithm 6.13 in the cases $(7, m+1)$, $W = [2i-1, n-2i-2] \cup [n-2i+1, n-2i+2]$ and $i > 1$, and $W = [3, n-1]$ with $i = 1$.

**Example 6.14.** Proof of Claim (7, m+1)(f) in the case $W = [2i-1, n-2i-2] \cup [n-2i+1, n-2i+2]$ and $i > 1$. Here $s = 2k + 2i - 1$ and $i \in [1, k/2]$ so $X_s = X_{2k+2i-1} = [2i-1, n-2i]$ and $i \in [2, k/2]$. Thus

$$B_{Z_2}^m - C_{W}^m = B_{X_s}^m - C_{W}^m = B_{[2i-1,n-2i]}^m - C_{[2i-1,n-2i-2]\cup[n-2i+1,n-2i+2]}^m.$$ 

After Step 1 we found that $Z_1 = Z \setminus [2i+2, n-2i-2] = [2i-1, 2i+1] \cup [n-2i-1, n-2i]$ and $W_1 = W \setminus [2i+2, n-2i-2] = [2i-1, 2i+1] \cup [n-2i+1, n-2i+2]$.

After Step 2 we find $Z_2 = Z_1$, $W_2 = W_1$ (since $[2k+1, 3k] \cap W_1 = \emptyset$).

Applying Step 3 we find

$$J_2 = \begin{cases} \emptyset, & \text{if } 2i + 1 < k \\ \{k\}, & \text{if } 2i + 1 = k \\ \{k-1, k\}, & \text{if } 2i = k. \end{cases}$$

So

$$Z_3 = \begin{cases} Z_2, & \text{if } 2i + 1 < k \\ Z_2 \setminus \{3k\} = [k - 2, k] \cup \{3k + 1\}, & \text{if } 2i + 1 = k \\ Z_2 \setminus \{3k - 1, 3k\} = [k - 1, k + 1], & \text{if } 2i = k \end{cases}.$$
and

\[ W_3 = \begin{cases} 
W_2, & \text{if } 2i + 1 < k \\
W_2 \setminus \{k\} = [k - 2, k - 1] \cup [3k + 2, 3k + 3], & \text{if } 2i + 1 = k \\
W_2 \setminus \{k, k + 1\} = \{k - 1, 3k + 1, 3k + 2\}, & \text{if } 2i = k.
\end{cases} \]

In all cases \( W_3 \subseteq [1, k - 1] \cup [3k + 1, n - 1] \) so Steps 4 and 5 leave \( W \) and \( Z \) unchanged; that is \( W_5 = W_3 \) and \( Z_5 = Z_3 \). Applying Step 6 we find

\[ P_5 = \begin{cases} 
\{2i - 1, 2i + 1, n - 2i + 2\}, & \text{if } 2i + 1 < k \\
\{k - 2, 3k + 3\}, & \text{if } 2i + 1 = k \\
\{k - 1, 3k + 2\}, & \text{if } 2i = k.
\end{cases} \]

Therefore

\[ Z_6 = Z_5 \setminus P_5 = \begin{cases} 
\{2i, n - 2i - 1, n - 2i\}, & \text{if } 2i + 1 < k \\
\{k - 1, k, 3k + 1\}, & \text{if } 2i + 1 = k \\
\{k, k + 1\}, & \text{if } 2i = k,
\end{cases} \]

\[ P_6 = P_5 \setminus Z_5 = \begin{cases} 
\{n - 2i + 2\}, & \text{if } 2i + 1 < k \\
\{3k + 3\}, & \text{if } 2i + 1 = k \\
\{3k + 2\}, & \text{if } 2i = k.
\end{cases} \]

and

\[ W_6 = W_5 \setminus P_5 = \begin{cases} 
\{2i, n - 2i + 1\}, & \text{if } 2i + 1 < k \\
\{k - 1, 3k + 2\}, & \text{if } 2i + 1 = k \\
\{3k + 1\}, & \text{if } 2i = k.
\end{cases} \]

Applying Step 7 we find

\[ Q_6 = \begin{cases} 
\{2i - 1, n - 2i\}, & \text{if } 2i + 1 < k \\
\{k - 2, 3k + 1\}, & \text{if } 2i + 1 = k \\
\{k - 1\}, & \text{if } 2i = k.
\end{cases} \]

So

\[ Z_7 = Z_6 \setminus Q_6 = \begin{cases} 
\{2i, n - 2i - 1\}, & \text{if } 2i + 1 < k \\
\{k - 1, k\}, & \text{if } 2i + 1 = k \\
\{k, k + 1\}, & \text{if } 2i = k,
\end{cases} \]

and

\[ Q_7 = Q_6 \setminus Z_6 = \begin{cases} 
\{2i - 1\}, & \text{if } 2i + 1 < k \\
\{k - 2\}, & \text{if } 2i + 1 = k \\
\{k - 1\}, & \text{if } 2i = k,
\end{cases} \]

that is,

\[ B^m_{Z_7} - B^m_{P_6} - B^m_{Q_7} = \begin{cases} 
B^m_{\{2i, n - 2i - 1\}} - B^m_{\{2i - 1, n - 2i + 2\}}, & \text{if } 2i + 1 < k \\
B^m_{\{k - 1, k\}} - B^m_{\{k - 2, 3k + 3\}}, & \text{if } 2i + 1 = k \\
B^m_{\{k, k + 1\}} - B^m_{\{k - 1, 3k + 2\}}, & \text{if } 2i = k.
\end{cases} \]
and we wish to show $B^m_{Z_1} - B^m_{P_6} - B^m_{Q_7} \geq 4B^m_{\{k\}}$.

Applying Step 8 gives

$$B^m_{Z_1} - B^m_{P_6} - B^m_{Q_7} = \begin{cases} B^m_{X_{2i}} + B^m_{X_{2i-2}} - B^m_{X_{2i+1}} - B^m_{X_{2i+2}}, & \text{if } 2i + 1 < k \\ B^m_{X_{k+1}} + B^m_{X_k} - B^m_{X_{k+2}} - B^m_{X_{k+3}}, & \text{if } 2i + 1 = k \\ B^m_{X_k} + B^m_{X_{k+1}} - B^m_{X_{k+2}} - B^m_{X_{k+3}}, & \text{if } 2i = k \end{cases}$$


$$= \begin{cases} B^m_{[2k-2i+2,2k+2i+1]} + B^m_{[2k-2i+1,2k+2i+2]} - B^m_{X_{[2k-2i+2,2k+2i+1]}} & \text{if } 2i + 1 < k \\ B^m_{[k+3,3k]} + B^m_{[k+1,3k]} - B^m_{[k+3,3k-2]} - B^m_{[k+4,3k-3]} & \text{if } 2i + 1 = k \\ B^m_{[k,3k-1]} + B^m_{[k+1,3k-1]} - B^m_{[k+2,3k-2]} - B^m_{[k+3,3k-2]} & \text{if } 2i = k \end{cases}$$

$$\geq 7B^m_{\{k\}} \quad \text{(by Lemma 6.2)}.$$

**Example 6.15.** Proof of Claim (7,m+1)(f) in the case $W = [3,n-1]$ and $i = 1$. Here $s = 2k + 1$ so $X_s = X_{2k+1} = [1,n-2]$. Therefore

$$B^m_Z - C^m_W = B^m_{X_s} - C^m_W = B^m_{[1,n-2]} - C^m_{[3,n-1]}.$$ 

After Step 1 we found that $Z_1 = Z \setminus [3,n-3] = \{1,2,n-2\}$ and $W_1 = W \setminus [3,n-3] = \{n-2,n-1\}$.

After Step 2 we find

$$Z_2 = \begin{cases} Z_1 = \{1,2,n-2\}, & \text{if } k > 2 \\ Z_1 \setminus \{n-2\} = \{1,2\}, & \text{if } k = 2 \end{cases}$$

and

$$W_2 = \begin{cases} W_1 = \{n-2,n-1\}, & \text{if } k > 2 \\ W_1 \setminus \{n-2\} = \{n-1\} (= \{7\}), & \text{if } k = 2. \end{cases}$$

In all cases $W_2 \subseteq [1,k-1] \cup [3k+1,n-1]$, so Steps 3, 4 and 5 leave $W$ and $Z$ unchanged; that is, $W_5 = W_2$ and $Z_5 = Z_2$.

Applying Step 6 we find

$$P_5 = \begin{cases} \{n-2\}, & \text{if } k > 2 \\ \emptyset, & \text{if } k = 2. \end{cases}$$

Thus $Z_6 = Z_5 \setminus P_5 = \{1,2\}$ in all cases, $P_6 = P_5 \setminus Z_5 = \emptyset$ in all cases and $W_6 = W_5 \setminus P_5 = \{n-1\}$ in all cases.

Applying Step 7 we find $Q_6 = \{1\}$, $Z_7 = Z_6 \setminus Q_6 = \{2\}$ and $Q_7 = Q_6 \setminus Z_6 = \emptyset$.

Thus

$$B^m_{Z_7} - B^m_{P_6} - B^m_{Q_7} = B^m_{Z_7} = B^m_{\{2\}}$$

in all cases, so we wish to show $B^m_{\{2\}} \geq 4B^m_{\{k\}}$. 

Applying Step 8,\[ B^m_{\{2\}} = \begin{cases} B^m_{\{2k,2k+3\}}, & \text{if } k > 2 \\ B^m_{\{2,5\}}, & \text{if } k = 2 \\ \geq 4B^m_{\{k\}} & \text{(by Lemma 6.2).} \]

**Proof of Claim (8, m+1).** By Claim (5, m+1)(c) and (d), \( \left| (B^{m+1})^{(2k-j)} \right| \geq \left| (C^{m+1})^{(2k-j)} \right| \) for \( j \in [1, k-1] \) unless both

\[ Y_{2k-j} = X_{2k-j} \text{ and } Y_{2k+j} = X_{2k+j}. \]

Also, by Claim (5, m+1) \( \left| (B^{m+1})^{(2k+j)} \right| \geq \left| (C^{m+1})^{(2k+j)} \right| \) for \( j \in [1, k-1] \).

Further, by Claim (6, m+1)(a) and (b)

\[ \left| (B^{m+1})^{(2k-j)} \right| + \left| (B^{m+1})^{(2k+j)} \right| \geq \left| (C^{m+1})^{(2k-j)} \right| + \left| (C^{m+1})^{(2k+j)} \right| \]

for \( j \in [1, k-1] \) provided both \( Y_{2k-j} = X_{2k-j} \) and \( Y_{2k+j} = X_{2k+j} \).

Combining these points we deduce

\[ (6.5) \quad \left| (B^{m+1})^{(2k-j)} \right| + \left| (B^{m+1})^{(2k+j)} \right| \geq \left| (C^{m+1})^{(2k-j)} \right| + \left| (C^{m+1})^{(2k+j)} \right| \]

for \( j \in [1, k-1] \).

Further, by Claim (5,m+1),

\[ \left| (B^{m+1})^{(k)} \right| \geq \left| (C^{m+1})^{(k)} \right|, \quad \left| (B^{m+1})^{(2k)} \right| \geq \left| (C^{m+1})^{(2k)} \right| \]

and

\[ \left| (B^{m+1})^{(3k)} \right| \geq \left| (C^{m+1})^{(3k)} \right| \]

Simple appropriate summation involving these latter four inequalities now establishes Claim (8, m+1).

**Proof of Claim (9, m+1).** Note that by symmetry it suffices to prove Claim (9, m+1) for \( j \in E[2, k-1] \cup O[3k + 1, n - 1] \).

**Case 1.** For \( j \in E[2, k-1] \), (that is, \( j = 2i \) for \( i \in [1, (k-1)/2] \)).

Note that \( \left| (B^{m+1})^{(n-j)} \right| \geq \left| (C^{m+1})^{(n-j)} \right| \) by Claim (5, m+1).

(a) If \( Y_{2i} \neq [2k - 2i, 2k + 2i - 1] \) then \( \left| (B^{m+1})^{(j)} \right| \geq \left| (C^{m+1})^{(j)} \right| \) by Claim (5, m+1)(a). Combining this with the above and with (6.5) establishes Claim (9, m+1).

(b) If \( Y_{2i} = [2k - 2i, 2k + 2i - 1] \), then by Corollary 4.13, for at least one \( s \in \{2k-2i, 2k+2i, n-2i\} \), \( Y_s \neq X_s \), and so by Claim (7, m+1), \( \left| (B^{m+1})^{(j)} \right| + \left| (B^{m+1})^{(s)} \right| \geq \left| (C^{m+1})^{(j)} \right| + \left| (C^{m+1})^{(s)} \right| \).

For \( s = n - 2i \), the above inequality and (6.5) establishes Claim (9, m+1).
For \( s \in \{2k-2i, 2k+2i\}, Y_s \neq X_s \) gives both \((B^{m+1})^{(2k-2i)} \geq (C^{m+1})^{(2k-2i)}\) and \((B^{m+1})^{(2k+2i)} \geq (C^{m+1})^{(2k+2i)}\), so the above inequality, one of these inequalities and the fact that \((B^{m+1})^{(n-2i)} \geq (C^{m+1})^{(n-2i)}\) establish Claim (9, m+1).

Case 2. For \( j \in O[3k+1, n-1]\), (that is, \( j = n - 2i + 1 \) for \( i \in [1, k/2]\)).

Note that \((B^{m+1})^{(n-j)} \geq (C^{m+1})^{(n-j)}\) by Claim (5, m+1).

(a) If \( Y_{n-2i+1} \neq [2k-2i+1, 2k+2i-2]\), \((B^{m+1})^{(j)} \geq (C^{m+1})^{(j)}\) by Claim (5, m+1)(b). Combining this with the above and with (6.5) establishes Claim (9, m+1).

(b) If \( Y_{n-2i+1} = [2k-2i+1, 2k+2i-2]\), then by Corollary 4.13, for at least one \( s \in \{2k-2i+1, 2k+2i-1, 2i-1\}, Y_s \neq X_s \), and so, by Claim (7, m+1),

\[
(B^{m+1})^{(j)} + (B^{m+1})^{(s)} \geq (C^{m+1})^{(j)} + (C^{m+1})^{(s)}.
\]

For \( s = 2i - 1 \), the above inequality and (6.5) establishes Claim (9, m+1).

For \( s = 2k - 2i + 1 \) or \( 2k + 2i - 1 \), \( Y_s \neq X_s \) gives both \((B^{m+1})^{(2k-2i+1)} \geq (C^{m+1})^{(2k-2i+1)}\) and \((B^{m+1})^{(2k+2i-1)} \geq (C^{m+1})^{(2k+2i-1)}\), so the above inequality together with one of these inequalities and the fact that \((B^{m+1})^{(2i-1)} \geq (C^{m+1})^{(2i-1)}\) establish Claim (9, m+1).

Proof of Claim (10, m+1). This follows directly from the fact that

\[
(B^{m+1})^{(k)} + (B^{m+1})^{(3k)} \geq (C^{m+1})^{(k)} + (C^{m+1})^{(3k)}
\]

and Claim (9, m+1) using summation.

Proof of Claim (11, m+1). This follows directly from Claim (8, m+1) and Claim (10, m+1) using summation.

This completes the proof of Lemma 6.4.

### 7. The remaining case

A simple corollary of Lemma 6.4 (xi) in the case \( j = 1 \), Notation 5.1, Lemma 5.4 and the comments following it, is that for \( n \equiv 0 \) (mod 4) the permutation \( \theta_n \) has maximum entropy amongst those \( n \)-cycles \( \phi \) which are maximodal and for which \( \phi(1) < \phi(2) \). If we can show that the entropy of \( \theta_n \) is at least as great as the entropy of any \( n \)-cycle \( \phi \) which is maximodal and for which \( \phi(1) > \phi(2) \), then by Theorem 2.10 and the paragraph immediately above it, \( \theta_n \) will have maximum entropy among all \( n \)-cycles \( \phi \). Theorem 4.2 will then be complete provided we can show \( \theta_n, \bar{\theta}_n \) and \( \theta_n^* = \theta_n \) are all cycles with the same entropy as \( \theta_n \). It turns out that \( \theta_n^* \) has a central role in the completion of our task.

**Lemma 7.1.** For \( i \in [1, n] \), \( \theta^*(i) = n + 1 - \theta(i) \).

**Lemma 7.2.** For a permutation \( \theta \in P_n \), \( n \) even, \( (\theta^*)^* = \theta \) and \( \theta^* \) is maximodal with \( \theta^*(1) > \theta^*(2) \) if and only if \( \theta \) is maximodal with \( \theta(1) < \theta(2) \).
**Lemma 7.3.** Let $\theta \in P_n$ and let $C$ be the induced matrix of $\theta$ and $D$ the induced matrix of $\theta^*$. Then for $i, j \in [1, n-1]$

$$c_{i,j} = 1 \iff d_{i,n-j} = 1.$$  

**Notation 7.4.** If $C$ is an $(n-1) \times (n-1)$ matrix whose only entries are 0 and 1 then $C^*$ is that $(n-1) \times (n-1)$ matrix given by

$$c_{i,j}^* = 1 \iff c_{i,n-j} = 1,$$

for all $i, j \in [1, n-1]$.

**Lemma 7.5.** If $C$ is an $(n-1) \times (n-1)$ matrix whose only entries are 0 and 1 then for all $p \in \mathbb{N} \cup \{0\}$ and all $j \in [1, n-1]$

$$\left| (C^p)_{(j)} \right| = \left| (C^p)_{(n-j)} \right|.$$

**Corollary 7.6.** If $C$ is an $(n-1) \times (n-1)$ matrix whose only entries are 0 and 1 then for all $p \in \mathbb{N} \cup \{0\}$

$$\|C^p\| = \|C^p\|.$$

**Corollary 7.7.** If $B^*$ is the induced matrix of $\theta^*_n$, then for all $p \in \mathbb{N} \cup \{0\}$,

$$\|B^p\| = \|B^p\|$$

and hence $\theta_n$ and $\theta^*_n$ have the same entropy.

**Notation 7.8.** If $\Gamma$ is the class of $(n-1) \times (n-1)$ matrices used in Lemma 6.4 then

$$\Gamma^* = \{C^* : C \in \Gamma\}.$$

**Corollary 7.9.** For all $C^* \in \Gamma^*$ and all $p \in \mathbb{N} \cup \{0\}$

$$\|C^p\| = \|C^p\| \leq \|B^p\| = \|B^p\|.$$

**Lemma 7.10.** Let $D$ be the induced matrix of an $n$-permutation $\phi$ such that $\phi$ is maximodal with $\phi(1) > \phi(2)$ and

(i) For some $i \in [1, (k-1)/2]$ the matrix $D$ is identical to the matrix $A^*$ on each of the four columns $2i, 2k - 2i, 2k + 2i, n - 2i$

or

(ii) For some $i \in [1, k/2]$ the matrix $D$ is identical to the matrix $A^*$ on each of the four columns $2i - 1, 2k - 2i + 1, 2k + 2i - 1, n - 2i + 1$.

Then $\phi$ is not a cycle.
Proof. First note that by Lemma 7.3, the matrix $D$ is identical to the matrix $A^*$ on column $j$ if and only if the matrix $D^*$ is identical to the matrix $A$ on column $n-j$, so (i) and (ii) above are equivalent to (i)' and (ii)' where (i)' and (ii)' are obtained from (i) and (ii) by replacing $D$ with $D^*$ and $A^*$ with $A$ throughout. A consequence of this is that $\phi^*$ satisfies precisely the conditions of Proposition 4.12, which establishes $\phi^*$ is not a cycle since for some $i \in [1, (k-1)/2], [2i+1, 2k+2i] \cup [2k+2i+1, n-2i]$ is fully invariant under $\phi^*$, or for some $i \in [1, k/2], [2i, 2k-2i+1] \cup [2k+2i, n-2i+1]$ is fully invariant under $\phi^*$. However $\phi = \rho \circ \phi^*$ where $\rho(j) = n+1-j$ for all $j \in [1, n]$ and the above mentioned sets are trivially fully invariant under $\rho$. Thus they are also fully invariant under $\phi$ and hence $\phi$ is not a cycle. \hfill \Box

Corollary 7.11. The set of all maximodal $n$-cycles $\phi$ for which $\phi(1) > \phi(2)$ is a subset of the set of all maximodal $n$-permutations $\theta$ for which $\theta(1) > \theta(2)$ and for which their induced matrices $M(\theta)$ satisfy the conditions

(i) For all $i \in [1, (k-1)/2]$ at least one of the following inequalities holds:

\[
M(\theta)(2i) \neq A^*(2i)
\]

\[
M(\theta)(2k-2i) \neq A^*(2k-2i)
\]

\[
M(\theta)(2k+2i) \neq A^*(2k+2i)
\]

\[
M(\theta)(n-2i) \neq A^*(n-2i)
\]

and

(ii) For all $i \in [1, k/2]$ at least one of the following inequalities holds:

\[
M(\theta)(2i-1) \neq A^*(2i-1)
\]

\[
M(\theta)(2k-2i+1) \neq A^*(2k-2i+1)
\]

\[
M(\theta)(2k+2i-1) \neq A^*(2k+2i-1)
\]

\[
M(\theta)(n-2i+1) \neq A^*(n-2i+1)
\]

Lemma 7.12. Let $\theta$ be a maximodal $n$-permutation for which $\theta(1) > \theta(2)$ and let $M(\theta)$ be its induced matrix. Let $M(\theta)$ satisfy conditions (i) and (ii) of Corollary 7.11. Then there exists an element $C^* \in \Gamma^*$ such that $C^*$ dominates $M(\theta)$.

Proof. The $n$-permutation $\theta^*$ is maximodal with $\theta^*(1) < \theta^*(2)$ and its induced matrix $M(\theta^*)$ satisfies condition 5 in the definition of $\Gamma$. Thus by the evident and valid strengthened version of Lemma 5.4, there is an element $C \in \Gamma$ such that $C$ dominates $M(\theta^*)$. But now $C^* \in \Gamma^*$ and by Corollary 7.6, $C^*$ dominates $M(\theta)$ as required. \hfill \Box

A combination of Corollaries 7.9 and 7.11 and Lemma 7.12 now completes our task of showing $\theta_n$ has maximum entropy among all $n$-cycles.

We have already seen the entropy of $\theta_n$ and the entropy of $\theta_n^*$ are identical. Combining this with the fact that taking duals is an entropy preserving idempotent operation shows $\theta_n$ and $\theta_n^*$ also have the same entropy. Finally, to show $\theta_n$, $\theta_n^*$ and $\theta_n^*$ are all cycles, note that $\theta_n(i) = j \iff \theta_n(n+1-i) = n+1-j$ (trivially the reason that $\theta$ is a cycle if and only if $\theta$ is a cycle), $\theta_n^*(i) = n+1-j \iff \theta_n(n+1-i) = j$, and that $\theta_n$ is a cycle implies $\theta_n$ is a cycle, $\theta_n$ is a cycle implies $\theta_n^*$ is a cycle.
References


