Natural Dualities for Quasi-varieties of Semigroups

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Contents

Summary

Pontryagin's duality for abelian groups and Stone duality for Boolean algebras were the seeds from which the theory of *natural dualities* grew. A natural duality can provide a useful method for representing every algebra in a quasi-variety as an algebra of continuous homomorphisms over some structured Boolean space. A finite algebra is *dualisable* if the quasi-variety it generates possesses a natural duality.

One of the fundamental open problems in the theory of natural dualities is 'Which finite algebras are dualisable?' In this thesis we investigate this problem in the class of semigroups. Motivated by issues that arise in the semigroup case, we also prove several results in the general theory.

We complete the characterisation of dualisability for finite normal bands: we find alter egos that dualise those finite normal bands that generate non-variety quasi-varieties. A Clifford semigroup is a semilattice of groups. We investigate dualisability within the class of Clifford semigroups that are semilattices of abelian groups. In addition, we show that a cyclic group of order m adjoined with a new identity element is dualisable.

It has been shown that the direct product of two dualisable algebras need not be dualisable. However, in this thesis we prove that the direct product of two dualisable algebras generating independent varieties is dualisable. Moreover, we show that if two finite algebras generate the same quasi-variety and one of them satisfies the Interpolation Condition (IC), then the other satisfies (IC).

Statement of authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis submitted for the award of any other degree or diploma.

No other person's work has been used without due acknowledgement in the main text of the thesis.

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Chapter 1

Introduction

Natural duality is a useful tool to study quasi-varieties generated by finite algebras. It translates algebraic problems to topological problems which may make them easier to solve. A natural duality for a quasi-variety provides us a uniform method for representing each algebra in the quasi-variety as the algebra of all continuous homomorphisms over some structured Boolean space. A finite algebra is *dualisable* if there is a natural duality for the quasi-variety it generates. One of the most difficult and open problems in the theory of natural dualities is the *dualisability problem*, which asks *'Which finite algebras are dualisable?'* This problem is in general and also in some familiar classes.

Duality theory has been studied since the 1930's. Stone [45] in his journey in understanding Boolean algebras has discovered the *first* natural duality between Boolean algebras and Boolean spaces, that much later led to a general theory of natural dualities. He discovered a representation for all Boolean algebras by using topological spaces and proved every Boolean algebra is isomorphic to the algebra of all clopen subsets of some totally disconnected compact space. In addition, he showed that the homomorphisms between Boolean algebras correspond to the continuous maps between the corresponding spaces. In the language of natural duality, he proved that the category of Boolean algebras is dually equivalent to the category of Boolean spaces.

A few years later, Birkhoff [4] and Pontryagin [37] discovered further dualities. Birkhoff showed that every finite distributive lattice is isomorphic to the lattice of all decreasing subsets of a finite ordered sets while Pontryagin discovered the duality for the category of abelian groups. These three results were the seeds for new dualities. A general notion of natural duality for a quasi-variety was initiated by Davey and Werner [16], generalizing well-known dualities such as Stone duality for Boolean algebras, Pontryagin duality for abelian groups, Priestley duality [38] for distributive lattices and Hofmann-Mislove-Stralka duality [27] for semilattices. More details on natural dualities may be found in Davey and Clark's text, *Natural Dualities for the Working Algebraist* which was the first text devoted to the study of natural dualities.

There has been some progress towards solving the dualisability problem for finite semigroups, especially within the classes of groups and bands (idempotent semigroups). It follows from Pontryagin [37] duality that a finite abelian group is dualisable. Quackenbush and Szabó [40] proved that every finite group with cyclic Sylow subgroups is dualisable. In the other direction, they showed that finite groups with non-abelian Sylow subgroups are not dualisable [39]. Apart from groups dualities, Davey and Knox worked on dualisability of bands and they proved that finite rectangular bands are dualisable [14]. In addition, in [13] they gave dualities for the quasi-varieties of left normal, right normal and normal bands. Dualities for semilattices, left-zero and right-zero semigroups are covered by Hofmann-Mislove-Stralka [27] and Banaschewski [3].

Jackson [31] has proved some results on dualisability of finite semigroups. He established the non-dualisability of a finite left-zero (right-zero) semigroup adjoined with an identity and some small semigroups including a commutative example. He showed that if a band is non-normal then it contains a subsemigroup isomorphic to a left-zero or right-zero semigroup adjoined with an identity. It follows that a finite band is contained in the quasi-variety of a finite dualisable algebra if and only if it is a normal band. Moreover, Jackson has shown that a finite inverse semigroup or a finite monoid is contained in the quasi-variety generated by a finite dualisable algebra if and only if it is a finite semilattice of groups with abelian Sylow subgroups under the assumption that Quackenbush and Szabó conjecture is true.

A variety is residually finite if and only if all of its subdirectly irreducible members are finite. A variety is called residually large if it has a proper class of subdirectly irreducible algebras. Golubov and Sapir [23] gave a description of all residually finite semigroups by three different ways: listing the semigroups that generate these varieties, determining the bases of identities defining these varieties, and characterizing the structure of an arbitrary semigroup from these varieties. McKenzie [35] classified (independently of [23]) residually finite varieties of semigroups. He showed that if a semigroup is not a group or not very closed to being union of groups, then it generates a residually large variety.

There is no obvious connection between dualisability of an algebra and the residual character of the variety it generates. It has been noticed that all known dualisable semigroups generate residually finite varieties. This thesis continues to reenforce this theme. However, it is not true that algebras that generate residually finite varieties are dualisable. For example, the two element implication algebra is non-dualisable, although it generates residually small variety.

In this thesis we investigate the dualisability of quasi-varieties of normal bands, some Clifford semigroups and the class of completely simple semigroups admitting natural dualities. We also prove some general results motivated by issues that arise in the semigroup case.

In Chapter 2, we give a brief background that will be needed in this thesis. We give an overview of some fundamental definitions and results on semigroup theory, quasi-varieties, natural duality theory and some dualities on semigroups.

Results in the literature classify most finite bands as either admitting a natural duality or being inherently non-dualisable. In Chapter 3, we complete the characterization of finite bands by showing that all of the remaining unclassified bands do admit a natural duality. We show that the direct product of a particular algebra with a 2-element right-zero semigroup with constant can be inherently non-dualisable. The results of this chapter are based on the paper Aldhamri [2].

In Chapter 4, we examine the special case of independent varieties and show that in this case, the direct product of two dualisable algebras is indeed dualisable. However, in general the direct product of two dualisable algebras need not be dualisable. The results in this chapter provide one half of the likely classification of dualisability for completely simple semigroups.

In Chapter 5, we prove that if two algebras generate the same quasi-variety and one of these algebras has an alter ego of finite type that satisfies the Interpolation Condition (IC), then the alter ego of the other algebra does as well. We can always find an alter ego of an algebra that satisfies (IC); simply by choosing the alter ego to be the brute-force alter ego (see The Brute Force Duality Theorem 2.3.1, [6]). The issue in this chapter is to take an alter ego of finite type for the first algebra and transfer it explicitly to an alter ego of finite type for the other algebra.

In Chapter 6, we investigate dualisability within the class of Clifford semigroups that are semilattices of abelian groups. We investigate dualisability of a cyclic group of order m adjoined with a new identity and describe some semigroups that generate the same quasi-varieties as this semigroup.

Finally, we establish the dualisability of a finite semigroup that is a semilattice of abelian groups of coprime orders.

Chapter 2

Preliminaries

In this chapter, we will give some fundamental definitions and results on semigroups, quasi-varieties, natural duality theory and present some dualities for semigroups that are needed in this thesis.

2.1 Semigroups

In this section we give elementary notions from semigroup theory. We refer to Howie's text [28]. A semigroup $\mathbf{M}$ is *regular* if, for all x in M, there exists y such that $xyx = x$. Groups are examples of regular semigroups. The semigroup $\mathbf{M}$ is called *completely regular* if there exists a unary operation $a \mapsto a^{-1}$ on $\mathbf{M}$ with the properties

$$(x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1} = x^{-1}x.$$

More briefly, a completely regular semigroup $\mathbf{M}$ is a semigroup in which every element of $\mathbf{M}$ lies in a subgroup of $\mathbf{M}$ or it is a union of groups. In this thesis we will not regard this unary operation as one of the fundamental operations. So, completely regular semigroups will be (mostly) treated in the signature consisting of the single binary operation of multiplication.

A semigroup $\mathbf{M}$ is *simple* if it has no proper ideal. Alternatively, a semigroup $\mathbf{M}$ is simple if and only if for every a, b in M there exists x, y in M such that $xay = b$. A *primitive* idempotent e is a non-zero idempotent element such that for every idempotent $f \in M$ we have

$$ef = fe = f \neq 0 \Rightarrow e = f.$$

A *completely simple* semigroup is a simple semigroup in which every idempotent is primitive. A finite simple semigroup is a completely simple semigroup.

A completely regular semigroup $\mathbf{M}$ is *completely simple* if for all $x, y \in M$, we have $xx^{-1} = (xyx)(xyx)^{-1}$. Completely simple semigroups admit a very powerful structure theorem [28, Theorem 3.3.1] which due to Rees and Suschkewitsch. The theorem showed that every completely simple semigroup $\mathbf{M}$ is isomorphic to a semigroup $\mathcal{M}[\mathbf{G}; I, \Lambda; P]$ where $\mathbf{G}$ is a group, I, Λ are non-empty sets, $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries from $\mathbf{G}$ and a multiplication on $\mathbf{M} = (I \times \mathbf{G} \times \Lambda)$ is defined by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu). \tag{$\star$}$$

The importance of completely simple semigroup is illustrated by the following theorem.

Theorem 2.1.1. [28, Theorem 4.1.3] *Every completely regular semigroup is a semilattice of completely simple semigroups.*

Groups and *bands* (semigroups satisfying $x^2 \approx x$) are examples of completely regular semigroups. The class of bands that satisfy the identity $xyzt \approx xzyt$, is known as the class of *normal bands* $\mathcal{N}$. The lattice of quasivarieties of normal bands has been completely described by Gerhard and Shafaat [22]. More details on normal bands will be given in Chapter 3. An important type of normal band is a *rectangular band* which is defined by the identity $xyx \approx x$. Every rectangular band is isomorphic to $\mathcal{M}[\mathbf{G}; I, \Lambda; P]$ where $\mathbf{G}$ is the trivial group. Then the $\mathcal{M}[\mathbf{G}; I, \Lambda; P]$ notation can be simplified to a structure on $I \times \Lambda$, with the product in Equation ($\star$) simplifying to

$$(i, \lambda)(j, \mu) = (i, \mu).$$

A such rectangular band on $I \times \Lambda$ is isomorphic to the direct product of a left-zero semigroup on I and a right-zero semigroup on Λ. Applying Theorem 2.1.1 to the class of bands, we get the following theorem.

Theorem 2.1.2. [28, Theorem 4.4.1] *Every band is a semilattice of rectangular bands.*

Completely regular semigroups satisfying the identity

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$$

are known as *Clifford semigroups.* Clifford semigroups are precisely the semilattices of groups. An *inverse semigroup* is a regular semigroup in which idempotents commute. Equivalently, the class of inverse semigroups is specified by the existence of a unary operation, $^{-1}$, satisfying

$$(x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad (xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

In the signature $\{\cdot, ^{-1}\}$ these form a variety, but in the plain semigroup signature they do not. An inverse semigroup is not completely regular unless it is a Clifford semigroup. The smallest inverse semigroup that is not completely regular is the Brandt semigroup $\mathbf{B}_2 = \mathcal{M}[G; I, I, \triangle]$ where $\mathbf{G}$ is a trivial group, $I = \{1, 2\}$ and $\triangle$ is the $I \times I$ diagonal matrix over group

G. Up to isomorphism, $\mathbf{B}_2$ is given by the following table.

$*$	0	a	b	c	a
0	0	0	0	0	0
a	0	0	c	0	a
b	0	d	0	b	0
c	0	a	0	c	0
d	0	0	b	0	d

2.2 Quasi-varieties

Consider a finite algebra $\mathbf{M} = \langle M; F\rangle$ of type F (not necessarily a semigroup). Let $\mathcal{K}$ be a class of algebras of the same type. The standard operators are defined as follows:

- $\mathbb{I}(\mathcal{K})$ is the class of all isomorphic copies of algebras in $\mathcal{K}$;
- $\mathbb{H}(\mathcal{K})$ is the class of all homomorphic copies of algebras in $\mathcal{K}$;
- $\mathbb{S}(\mathcal{K})$ is the class of all subalgebras of algebras in $\mathcal{K}$;
- $\mathbb{P}(\mathcal{K})$ is the class of all products of algebras in $\mathcal{K}$.

An *equation* is an expression of the form $t_1 \approx t_2$ where t_1 and t_2 are terms. An *implication* is an expression of the form

$$p_1 \approx q_1 \ \& \ \cdots \ \& \ p_n \approx q_n \Rightarrow p_0 \approx q_0$$

where $p_i \approx q_i$ are equations. A *quasi-equation* is an expression that is either equation or implication. The *variety* is a class of algebras of the same type defined by a set of equations while a *quasi-variety* is a class of algebras of the same type defined by some set of quasi-equations. A variety may equivalently be defined as a class of algebras of the same type closed under the operators $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. A quasi-variety that admits a finite basis for its quasi-equations is said to be *finitely based*.

Let X be a set of maps from a set A to a set B and let C be a subset of A. The set X *separates the elements* of C if, for each $a, b \in C$ where $a \neq b$, there is a map $x_{ab}: A \to B$ in X such that $x_{ab}(a) \neq x_{ab}(b)$. In particular, we say an algebra $\mathbf{M}$ is *separated by homomorphisms* into an algebra $\mathbf{N}$ if the set of all homomorphisms from $\mathbf{M}$ to $\mathbf{N}$ separates the elements of $\mathbf{M}$.

The Algebraic Separation Theorem 2.2.3 describes the quasi-variety $\mathcal{A}$ generated by a finite algebra $\mathbf{M}$, it shows that the members of $\mathcal{A}$ are characterized by their homomorphisms into $\mathbf{M}$. This theorem will be used without reference in most chapters.

Algebraic Separation Theorem 2.2.1. (See [6, Theorem 1.3.1]) *An algebra $\mathbf{A}$ is in $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ if and only if, for each $a, b \in A$ where $a \neq b$, there is a separating homomorphism $x_{ab}: \mathbf{A} \to \mathbf{M}$ such that $x_{ab}(a) \neq x_{ab}(b)$.*

Example 2.2.2. *By the Fundamental Theorem of Abelian Groups, every finite abelian $\mathbf{G}$ generates the same quasi-variety as a cyclic group $\mathbf{C}_m$ for some m, that is, $\mathbb{ISP}(\mathbf{G}) = \mathbb{ISP}(\mathbf{C}_m)$.*

The following theorem describes the quasi-variety generated by a finite algebra.

Theorem 2.2.3. (See [36, Theorem 1.1.1], [6, Theorem 3.4]) *Let $\mathbf{M}$ be a finite algebra. For every algebra $\mathbf{A}$ of the same type as $\mathbf{M}$, the following are equivalent:*

(1) *$\mathbf{A}$ is obtainable from $\mathbf{M}$ by repeated applications of $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}$;*

(2) *$\mathbf{A}$ is separated by homomorphisms into $\mathbf{M}$;*

(3) *$\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$;*

(4) *$\mathbf{A}$ satisfies all quasi-equations satisfied by $\mathbf{M}$.*

In particular, the quasi-variety generated by $\mathbf{M}$ is the class $\mathbb{ISP}(\mathbf{M})$.

2.3 Topological Structures

A topological space is a *Boolean space* if it is both compact and totally disconnected. Notice that every finite discrete space is a Boolean space. A *natural duality* provides a dual representation between a finitely generated quasi-variety $\mathcal{A}$ and a category $\mathcal{X}$ of structured Boolean spaces. For example, Stone duality shows that the quasi-variety of Boolean algebras is dually equivalent to the category of Boolean spaces. In this section, we give the definition of a topological structure and of a structured Boolean space.

We start with three sets of symbols. Let G be a set of finitary total operation symbols; these may be nullary. Let H be a set of finitary partial operation symbols and R be a set of finitary relation symbols. Partial operations and relations must have positive arities. A *structured topological space* of type $\langle G, H, R\rangle$ is a structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$, where

(i) $G^{\mathbf{X}}$ consists of an n-ary total operation $g^{\mathbf{X}}\colon X^n \to X$ for each n-ary total operation symbol $g \in G$,

(ii) $H^{\mathbf{X}}$ consists of an n-ary partial operation $h^{\mathbf{X}}\colon\ \mathrm{dom}(h^{\mathbf{X}}) \to X$ for each n-ary partial operation symbol $h \in H$, where $\mathrm{dom}(h^{\mathbf{X}}) \subseteq X^n$,

(iii) $R^{\mathbf{X}}$ consists of an n-ary relation $r^{\mathbf{X}} \subseteq X^n$ on X for each n-ary relation symbol r,

(iv) $\langle X; \mathcal{T}^{\mathbf{X}}\rangle$ is a topological space.

In this thesis, all topological spaces will be Boolean spaces, every operation is continuous and each partial operation has closed domain (and is continuous from this domain). Indeed, in most concrete cases topologies on finite spaces always be discrete. For the class of topological structure $\mathcal{W}$ and a subclass $\mathcal{Y}$ of $\mathcal{W}$, the standard operators are defined as follows:

- $\mathbb{I}(\mathcal{Y})$ is the class of isomorphic copies (within $\mathcal{W}$);

- $\mathbb{S}_c(\mathcal{Y})$ is the class of all topologically closed substructures of $\mathcal{Y}$;
- $\mathbb{P}^+(\mathcal{Y})$ is the class of all direct products over *non-empty* index sets of members of $\mathcal{Y}$.

Let $\mathbf{Y} = \langle Y; G^{\mathbf{Y}}, H^{\mathbf{Y}}, R^{\mathbf{Y}}, \mathfrak{T}^{\mathbf{Y}} \rangle$ be another structured topological space. The structure $\mathbf{Y}$ is a *substructure* of the structure $\mathbf{X}$, written $\mathbf{Y} \leqslant \mathbf{X}$, if $Y \subseteq X$ and

(i) for each n-ary $g \in G$ the operation $g^{\mathbf{Y}}$ and operation $g^{\mathbf{X}}$ agree on Y^n,

(ii) for each n-ary $h \in H$, we have $\mathrm{dom}(h^{\mathbf{Y}}) = \mathrm{dom}(h^{\mathbf{X}}) \cap Y^n$, and the operation $h^{\mathbf{Y}}$ agrees with $h^{\mathbf{X}}$ on this set (in particular, for $a \in \mathrm{dom}(h^{\mathbf{Y}})$ we have $h^{\mathbf{X}}(a) \in Y$),

(iii) for each n-ary $r \in R$, we have $r^{\mathbf{Y}} = r^{\mathbf{X}} \cap Y^n$, and

(iv) $\mathfrak{T}^{\mathbf{Y}}$ is the relative topology obtained from $\mathfrak{T}^{\mathbf{X}}$.

A continuous map $\varphi\colon X \to Y$ between two structures $\mathbf{X}, \mathbf{Y}$ is a *morphism* if it preserves each member of $G \cup H \cup R$. More specifically, the map φ is a morphism if

(i) for each n-ary $g \in G$ and each $(x_1, \dots, x_n) \in X^n$, we have

$$\varphi(g^{\mathbf{X}}(x_1, \dots, x_n)) = g^{\mathbf{Y}}(\varphi(x_1), \dots, \varphi(x_n)),$$

(ii) for each n-ary $h \in H$ and each $(x_1, \dots, x_n) \in \mathrm{dom}(h^{\mathbf{X}})$, we have

$$(\varphi(x_1), \dots, \varphi(x_n)) \in \mathrm{dom}(h^{\mathbf{Y}}) \text{ and}$$
$$\varphi(h^{\mathbf{X}}(x_1, \dots, x_n)) = h^{\mathbf{Y}}(\varphi(x_1), \dots, \varphi(x_n)),$$

(iii) for each n-ary $r \in R$ and each $(x_1, \dots, x_n) \in r^{\mathbf{X}}$, we have

$$(\varphi(x_1), \dots, \varphi(x_n)) \in r^{\mathbf{Y}}.$$

A morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is called *isomorphism* if there is a morphism $\psi\colon Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. A morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is an *embedding* of $\mathbf{X}$ into $\mathbf{Y}$ if the image $\varphi(\mathbf{X})$ forms a substructure of $\mathbf{Y}$and φ is an isomorphism between $\mathbf{X}$ and $\varphi(\mathbf{X})$. The structure $\mathbf{Z}$ is *injective* in the class $\mathcal{X}$ if $\mathbf{Z} \in \mathcal{X}$ and for every morphism $\alpha\colon \mathbf{X} \to \mathbf{Z}$ and embedding $\varphi\colon \mathbf{X} \to \mathbf{Y}$ in $\mathcal{X}$, there is a morphism $\beta\colon \mathbf{Y} \to \mathbf{Z}$ such that $\beta \circ \varphi = \alpha$.

Note that when there is no confusion, we will omit the superscripts on $G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}$ and their members for any structure $\mathbf{X}$.

Let $\{\mathbf{Y}_i \mid i \in I\}$ be a family of finite discrete spaces and let $\mathbf{Z} = \prod_{i\in I} \mathbf{Y}_i$ be the Boolean space produced by endowing $\prod_{i\in I} \mathbf{Y}_i$ with the product topology. Let $X \subseteq Z$, we say that a map $\varphi\colon X \to Y$ has a *finite support* if there are a finite set $J \subseteq I$ and a map $\psi\colon \pi_J(X) \to Y$ such that $\varphi = \psi \circ \pi_J{\restriction}_X$, where π_J is the restriction to the set J. These spaces are so fundamental to natural dualities, the following lemma (taken directly from [6, B.6 Lemma]) gives an explicit description of their properties.

Lemma 2.3.1. [6, B.6 Lemma] *Let $\mathbf{Z} = \prod_{i\in I} \mathbf{Y}_i$ be the product of a non-empty collection of finite discrete spaces $\{\mathbf{Y}_i \mid i \in I\}$. For each $i \in I$ and each $a \in Y_i$ define $U_{i,a} = \{x \in Z \mid x(i) = a\}$.*

(1) *The set $\mathcal{S} = \{U_{i,a} \mid i \in I; a \in Y_i\}$ is a clopen subbasis for the topology on $\mathbf{Z}$.*

(2) *A subset $U \subseteq Z$ is clopen if and only if there is a finite subset $J \subseteq I$ and a subset $V \subseteq \prod_{j\in J} \mathbf{Y}_i$ such that U is the set of all members of Z whose restriction to J is in V.*

(3) *The closure of a subset $X \subseteq Z$ consists of all $y \in Z$ which agree with a member of X on each finite subset of I.*

(4) *If $\mathbf{Y}$ is a finite discrete space and $\mathbf{X} \subseteq \mathbf{Z}$ is closed, then a map $\varphi\colon X \to Y$ is continuous if and only if it has finite support.*

2.4 Natural Duality

The theory of natural duality provides a tool for understanding a quasi-variety generated by a finite algebra. In this section, we give a brief background of natural dualities and refer to the Clark and Davey text [6] for details. Let $\mathbf{M}$ be a finite algebra and let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a discrete topological structure where

(i) G is a set of total operations on M such that for $g \in G$ of arity $n \geq 1$ $g\colon \mathbf{M}^n \to \mathbf{M}$ is a homomorphism;

(ii) H is a set of partial operations on M such that if $h \in H$ is n-ary then the domain, $\mathrm{dom}(h)$, of h is a (non-empty) subalgebra of $\mathbf{M}^n$ and $h\colon\ \mathrm{dom}(h) \to \mathbf{M}$ is a homomorphism;

(iii) R is a set of finitary relations on M such that if $r \in R$ is n-ary then r forms a subalgebra of $\mathbf{M}^n$;

(iv) $\mathcal{T}$ is the discrete topology on M.

Then the structure $\underset{\sim}{\mathbf{M}}$ is known as an *alter ego* of $\mathbf{M}$. Under these conditions, there is a naturally defined dual adjunction between the quasi-variety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ and the topological quasi-variety $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ (consisting of isomorphic copies of possibly empty topologically closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$), as follows:

(i) for each $\mathbf{A} \in \mathcal{A}$ the homset $D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \mathbf{M})$ (which is the set of homomorphisms $\mathbf{A} \to \mathbf{M}$) is a closed substructure of $\underset{\sim}{\mathbf{M}}^A$;

(ii) for each $\mathbf{X} \in \mathcal{X}$ the homset $E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ (which is the set of continuous maps $X \to M$ that preserve each total operation, partial operation and relation in $G \cup H \cup R$) forms a subalgebra of $\mathbf{M}^X$;

(iii) for each $\mathbf{A} \in \mathcal{A}$, the evaluation map $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ defined for each $a \in A$ by

$$e_{\mathbf{A}}(a)(u) := u(a),$$

for each $u \in D(\mathbf{A})$, is an embedding;

(iv) for each $\mathbf{X} \in \mathcal{X}$, the evaluation $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$ defined for each $x \in \mathbf{X}$ by

$$\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x),$$

for each $\alpha \in E(\mathbf{X})$, is an embedding.

If $e_{\mathbf{A}}$ is an isomorphism for each algebra $\mathbf{A} \in \mathcal{A}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a *natural duality* on $\mathcal{A}$. We say $\mathbf{M}$ is *dualisable* if there exists some structure $\underset{\sim}{\mathbf{M}}$ which yields a duality on $\mathcal{A}$, in which it is often simply said that $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$) dualises $\mathbf{M}$. If $\varepsilon_{\mathbf{X}}$ is an isomorphism for each algebra $\mathbf{X} \in \mathcal{X}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a *full duality* on $\mathcal{A}$. In this thesis, we will not be considering full duality. The following theorem gives an alternative definition of dualisable algebra $\mathbf{M}$.

First Duality Theorem 2.4.1. [6, Theorem 2.2] *The following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}$*;*

(2) *for all* $\mathbf{A} \in \mathcal{A}$*, every morphism* $\alpha \colon D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *extends to an* A*-ary term function* $t \colon M^A \to M$ *on* $\mathbf{M}$*;*

(3) *the following two conditions hold:*

(INJ) $\underset{\sim}{\mathbf{M}}$ *is injective with respect to those embeddings in* $\mathcal{X}$ *that are of the form* $D(u) \colon D(\mathbf{A}) \to D(\mathbf{B})$ *where* $u \colon \mathbf{B} \to \mathbf{A}$ *is a surjective homomorphism, that is, for each morphism* $\alpha \colon D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *there exists a morphism* $\beta \colon D(\mathbf{B}) \to \underset{\sim}{\mathbf{M}}$ *such that* $\beta \circ D(u) = \alpha$*,* (CLO) *for each $n \in \mathbb{N}$, every morphism $t\colon \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ is an n-ary term function on* $\mathbf{M}$.

If $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and G, H and R are finite, then we say that $\underset{\sim}{\mathbf{M}}$ is of *finite type*. Willard [47] and Zádori [48] proved the following theorem independently.

Duality Compactness Theorem 2.4.2. [6, Theorem 2.11] *If $\underset{\sim}{\mathbf{M}}$ is of finite type and yields a duality on each finite algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, then $\underset{\sim}{\mathbf{M}}$ dualises* $\mathbf{M}$.

(IC) Duality Theorem 2.4.3. (See [6, Pages 51–52]) *Suppose $G \cup H \cup R$ is finite. Then $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$ provided the following* ***interpolation condition*** **(IC)** *is satisfied:*

(IC) *for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$, every morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to a term function $t\colon M^n \to M$ of the algebra* $\mathbf{M}$.

The condition (CLO) in Theorem 2.4.1 together with the strengthen of the condition (INJ) at the finite level are equivalent to the Interpolation Condition (IC). Davey and Willard [17] and Saramago [43] proved independently the following fundamental result.

Independence of Generator Theorem 2.4.4. (See [17], [43]) *Let $\mathbf{M}$ and $\mathbf{N}$ be finite algebras. If $\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{N})$, then $\mathbf{M}$ is dualisable if and only if $\mathbf{N}$ is dualisable.*

We conclude this section by looking briefly at non-dualisable algebras. The 2-element implication algebra $\mathbf{I} = \langle\{0,1\}; \to\rangle$ is the most well-known example of non-dualisable algebra [16, Pages 148–151]. By adding the nullary operation 1 to signature of $\mathbf{I}$, it becomes term equivalent to the 2-element Boolean algebra which is dualisable by Stone duality. The following non-dualisable algebra example is derived from the implication algebra.

Example 2.4.5. [12, Example 6.3] *Let* $\mathbf{I}$ *be the* 2*-element implication algebra. For* $a, b \in \{0, 1\}$*, define* $\mathbf{I}_{a,b} = \langle \{0, 1\}, \rightarrow, a, b\rangle$*. Then* $\mathbf{I}_{0,1}$ *and* $\mathbf{I}_{1,0}$ *are dualisable but* $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ *is not.*

A finite algebra $\mathbf{N}$ is called *inherently non-dualisable* (abbreviated to IND) provided $\mathbf{M}$ is non-dualisable whenever $\mathbf{M}$ is a finite algebra with $\mathbf{N} \in \mathbb{ISP}(\mathbf{M})$, or equivalently, if $\mathbf{M}$ is non-dualisable whenever $\mathbf{N}$ embeds into $\mathbf{M}$. The concept of inherently non-dualisable was introduced by Davey, Idziak, Lampe and McNulty [11]. We recall the Inherently Non-Dualisable Algebra Theorem which will be required in Chapter 3.

Inherently Non-dualisable Theorem 2.4.6. [6, Theorem 10.5.5] *Let* $\mathbf{N}$ *be a finite algebra. Assume that there exists an infinite set* S*, a subalgebra* $\mathbf{A}$ *of* $\mathbf{N}^S$ *and an infinite subset* A_0 *of* A *such that*

(1) *there is a function* $u \colon \mathbb{N} \rightarrow \mathbb{N}$ *such that if* θ *is a congruence on* $\mathbf{A}$ *of finite index at most* n*, then* $\theta{\restriction}_{A_0}$ *has only one class with more than* $u(n)$ *elements,*

(2) $g \notin A$*, where* g *is the element of* N^S *such that* $g(s) := \pi_s(b)$*, for each* $s \in S$*, with* b *any element of the block of* $\ker(\pi_s){\restriction}_{A_0}$ *which has size greater than* $u(|N|)$*.*

Then $\mathbf{N}$ *is inherently non-dualisable.*

The element g in item (2) is referred as the *ghost element*.

2.5 Duality results on semigroups

There has been some work on dualisability of semigroups, especially bands and groups. It follows from Pontryagin duality that every finite abelian group is dualisable. Davey and Quackenbush [15] proved that the finite dihedral groups D_n, for odd n, are dualisable. Moreover, Quackenbush and Szabó [40] showed that a finite group with cyclic Sylow subgroups is dualisable. In the other direction, they proved that finite non-abelian nilpotent groups are not dualisable [39]. In fact, their proof shows they are inherently non-dualisable. Sporadic examples of finite bands that are non-dualisable, have appeared as examples in several places. Hobby [26] has studied an infinite family of finite semigroups including some instances of bands and has shown that most of them are non-dualisable. This was extended by Jackson [31] who showed that a quasi-variety of bands cannot be dualisable if it does not consist of normal bands. The following are some examples of dualisable varieties of bands ($x^2 \approx x$ is implicit throughout these examples).

- The variety $\mathcal{S}$ of semilattices is defined by the identity $xy \approx yx$ and idempotence is already explicitly identified. The meet semilattice with 1, $\mathbf{S} = \langle \{0, 1\}; \wedge, 1 \rangle$ was proved to be dualisable by Hofmann, Mislove and Stralka [27] (for more details we also refer the reader to Davey and Werner [16]) and its alter ego is obtained by taking the existing operations and adding the discrete topology, so $\underset{\sim}{\mathbf{S}} = \langle \{0, 1\}; \wedge, 1, \mathcal{T} \rangle$. A simple modification of the proof shows that $\langle \{0, 1\}; \wedge \rangle$, $\langle \{0, 1\}; \wedge, 0 \rangle$, $\langle \{0, 1\}; \wedge, 0, 1 \rangle$ are also dualisable.

- The variety $\mathcal{L}$ of left-zero semigroups is defined by the identities $xy \approx x$ and $x^2 \approx x$ while the variety $\mathcal{R}$ of right-zero semigroups satisfies the dual identity. As these are term-equivalent to sets, duality for these two varieties is covered by the duality for sets given by Banaschewski [3] (for details see [16]).

- The variety $\mathcal{RB}$ of rectangular bands is defined by the identities $xyx \approx x$ and $x^2 \approx x$. Clark and Davey [6] gave a natural duality for the variety of rectangular bands while Davey and Knox [14] gave a new proof that every finite rectangular band is naturally dualisable.

- Davey and Knox [13] gave a sufficient condition for the dualisability of the quasi-variety generated by a finite dualisable algebra with added zero. As a result, the variety of left normal bands is naturally dualisable [13, Theorem 3.6].

Let **L** be the 2-element left-zero semigroup and **R** be the 2-element right-zero semigroup. The semigroups $\mathbf{L}^1$ and $\mathbf{R}^1$ are obtained by adjoining an identity element 1 to **L** and **R**, respectively. The possible dualisability of $\mathbf{L}^1$ and $\mathbf{R}^1$ was an open question in [26] until Jackson [31] proved that they are IND. Jackson [31] showed that $\mathbf{L}^1$ is inherently non-dualisable. For completeness we present the following proof of Lemma 2.5.2 to illustrate how the Inherently Non-dualisable Theorem 2.4.6 is used. This will be needed to prove a result in Chapter 3. We shall first introduce the following notation.

Notation 2.5.1. Let $a, a_1, a_2, \ldots, a_n$ be elements of a finite set M and let $i_1, i_2, \ldots, i_n \in \mathbb{N}$. Then, we let $a^{a_1\ a_2\ \cdots\ a_n}_{i_1\ i_2\ \cdots\ i_n}$ denote the element of $M^{\mathbb{N}}$ defined by

$$a^{a_1\ a_2\ \cdots\ a_n}_{i_1\ i_2\ \cdots\ i_n}(k) = \begin{cases} a_j & \text{if } k = i_j, \\ a & \text{otherwise.} \end{cases}$$

Let $\mathbf{M} = \langle \{a, b, c\}; * \rangle$ be a semigroup and assume that $\{a, b\}$ forms a 2-element right-zero subsemigroup of **M** under $*$, $cb \neq a$, $ac = a$ and $bc = b$. (As shown in the following table.)

$*$	a	b	c
a	a	b	a
b	a	b	b
c	?	$\neq a$	?

It is clear that the semigroup $\mathbf{R}^1$ is a such an algebra.

Lemma 2.5.2. [31] **M** *is inherently non-dualisable.*

Proof. We will use Theorem 2.4.6. Let S be an infinite set and let $\mathbf{A}$ be the subsemigroup of $\mathbf{M}^S$ generated by the set $\{h \mid h(i) = b, \text{ for some } i \in \mathbb{N}\}$. Let $A_0 = \{a_i^b \mid i \in \mathbb{N}\}$. It is clear that $A_0 \subseteq A$. Let $u \colon \mathbb{N} \to \mathbb{N}$ be the function with $u(n) = 1$, for all n and let θ be a congruence on $\mathbf{A}$ with index n. Assume that $a_i^b \;\theta\; a_j^b$ and $a_k^b \;\theta\; a_l^b$ with i, j, k, l pairwise unequal. We aim to show that $a_j^b \;\theta\; a_k^b$ and therefore there is a unique block of $\theta{\restriction}_{A_0}$ with more than $u(n)$ elements. Now we have $a_j^b * a_{j\,k}^{c\,b} = a_{j\,k}^{b\,b}$ and $a_i^b * a_{j\,k}^{c\,b} = a_k^b$ and hence $a_{j\,k}^{b\,b} \;\theta\; a_k^b$. Then by symmetry we have $a_j^b \;\theta\; a_{k\,j}^{b\,b} = a_{j\,k}^{b\,b} \;\theta\; a_k^b$ as required.

Now, let π_i be a projection map. It is clear that $\pi_i(a_j^b) = a$ whenever $i \neq j$. Hence we have $g(i) = a$ for all $i \in \mathbb{N}$ and so $g = \underline{a}$. We will show that $\underline{a}$ does not belong to the semigroup $\mathbf{A}$. If $h := h_1 * h_2 * \cdots * h_n$ is a product of generators in $\mathbf{A}$, then there exist i such that $h_n(i) = b$. Since $xb \neq a$, for all $x \in M$, then

$$h(i) = h_1(i) * h_2(i) * \cdots * h_n(i) = h_1(i) * h_2(i) * \cdots * b \neq a.$$

Hence, the ghost element $g = \underline{a}$ does not belong to $\mathbf{A}$. Then $\mathbf{M}$ is inherently non-dualisable. □

Jackson [31] also showed that if $\mathbf{B}$ is a non-normal band then $\mathbf{B}$ contains a subsemigroup isomorphic to $\mathbf{L}^1$ or $\mathbf{R}^1$. As the quasi-variety of all normal bands is generated as a quasi-variety by the semigroup obtained by adjoining a zero element to the rectangular band $\mathbf{L} \times \mathbf{R}$ which is dualisable [13], then we get the following result.

Theorem 2.5.3. [31] *A finite band is inherently non-dualisable if and only if it is not a normal band.*

As consequence of this theorem, to complete the dualisability classification of finite bands, it is sufficient to consider normal bands. We complete this classification in Chapter 3.

Chapter 3

Dualities for quasi-varieties of Normal Bands

Normal bands were first studied by McLean [34]. Kimura [33] described all identities on bands which have at most three variables, but a complete classification of band varieties was given independently in the early 1970s by Fennemore [19], Gerhard [21] and Biryukov [5], each giving a full description of the lattice of band varieties. Gerhard and Shafaat [22] completed the description of the lattice of quasi-varieties of normal bands.

There has been some progress towards characterizing dualisability within the class of normal bands. Davey and Knox worked on dualisability of bands and they proved that finite rectangular bands are dualisable [14]. In addition, in [13] they gave dualities for the quasi-varieties of left normal, right normal and normal bands. Dualities for semilattices, left-zero and right-zero semigroups are covered by Hofmann-Mislove-Stralka [27] and Banaschewski [3]. Jackson [31] has shown that a quasi-variety of bands cannot admit a natural duality if it does not consist of normal bands. In this chapter, we complete the classification of the dualisability of finite bands by showing that all of the remaining unclassified normal bands do admit a natural duality.

3.1 Preliminaries

In the following description, we refer to Howie's text [28]. The lattice of subvarieties of the variety $\mathcal{N}$ of normal bands is composed of eight varieties as shown in Figure 3.1. The atoms are the varieties $\mathcal{L}, \mathcal{S}, \mathcal{R}$ of left-zero semigroups, semilattices and right-zero semigroups, respectively. The varieties left normal bands $\mathcal{L}^0$, right normal bands $\mathcal{R}^0$ and rectangular bands $\mathcal{RB}$ are the remaining nontrivial, proper subvarieties.

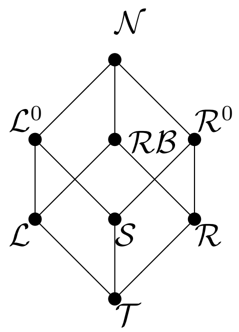

Figure 3.1: Varieties of Normal Bands

Shafaat [44] showed that the lattices of varieties and quasi-varieties have the same set of atoms: the set of varieties $\{\mathcal{L}, \mathcal{S}, \mathcal{R}\}$. He described the lattice of quasi-varieties of normal bands by the following theorem.

Theorem 3.1.1. [44, Theorem 4] *The following is a complete list of quasi-varieties of normal bands (and quasi-equations defining them within $\mathcal{N}$):*

(1) $\mathcal{T}$*:* $[x = y]$*;*

(2) $\mathcal{L}$*:* $[xy = x]$*;*

(3) $\mathcal{S}$*:* $[xy = yx]$*;*

(4) $\mathcal{R}$*:* $[xy = y]$*;*

(5) $\mathcal{L} \vee \mathcal{S}$*:* $[xz = yz \to xy = yx]$*;*

(6) $\mathcal{RB}$*:* $[xyx = x]$*;*

(7) $\mathcal{S} \vee \mathcal{R}$*:* $[zx = zy \to xy = yx]$*;*

(8) $\mathcal{RB} \vee \mathcal{S}$: $[xzy = yzx \to xy = yx]$;

(9) $\mathcal{L}^0$: $[xyz = xzy]$;

(10) $\mathcal{R}^0$: $[xyz = yxz]$;

(11) $\mathcal{L}^0 \vee \mathcal{R}$: $[zx = zy \to uxy = uyx]$;

(12) $\mathcal{R}^0 \vee \mathcal{L}$: $[xz = yz \to xyu = yxu]$;

(13) $\mathcal{N}$: $[xyzx = xzyx]$.

The lattice is depicted in Figure 3.2, where solid points depict varieties.

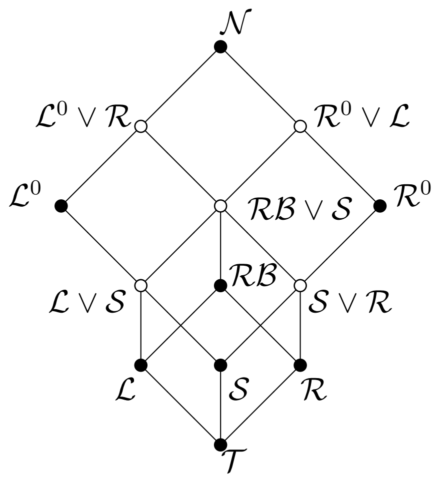

Figure 3.2: Quasi-varieties of Normal Bands

It is straight forward to verify that these quasi-varieties are generated by basic semigroups as follows:

- $\mathcal{L} = \mathbb{ISP}(\mathbf{L})$;
- $\mathcal{S} = \mathbb{ISP}(\mathbf{S})$;
- $\mathcal{R} = \mathbb{ISP}(\mathbf{R})$;
- $\mathcal{RB} = \mathbb{ISP}(\mathbf{L} \times \mathbf{R})$;
- $\mathcal{L}^0 = \mathbb{ISP}(\mathbf{L}^0)$;

- $\mathcal{L} \vee \mathcal{S} = \mathbb{ISP}(\mathbf{L} \times \mathbf{S})$;
- $\mathcal{RB} \vee \mathcal{S} = \mathbb{ISP}(\mathbf{RB} \times \mathbf{S})$;
- $\mathcal{L}^0 \vee \mathcal{R} = \mathbb{ISP}(\mathbf{L}^0 \times \mathbf{R})$;
- $\mathcal{N} = \mathbb{ISP}(\mathbf{L}^0 \times \mathbf{R}^0)$.

Let $L = S = R = \{0, 1\}$, let $L^z = \{0, 1, z\}$ and let $RB = L \times R$ be the underlying sets of the left-zero semigroup $\mathbf{L}$, the meet semilattice $\mathbf{S}$, the right-zero $\mathbf{R}$, the left normal semigroup $\mathbf{L}^z$ obtained by adjoining a zero z to $\mathbf{L}$, and the 4-element rectangular band $\mathbf{RB}$ isomorphic to product of $\mathbf{L}$ and $\mathbf{R}$, respectively. It is straight forward to verify that the list above shows that each quasi-variety of normal bands is generated by one of the semigroups $\mathbf{L}, \mathbf{S}, \mathbf{R}, \mathbf{L}^0$ and $\mathbf{R}^0$ (and is a variety) or by a direct product of some combinations of these.

3.2 Dualities for Quasi-varieties of Normal Bands

In this section we will prove that the five nonvariety quasi-varieties shown in Figure 3.2 are all dualisable by showing that they possess alter egos satisfying (IC). Notice that from the previous list that each of these five quasi-varieties is generated by a direct product of some pair of the bands $\mathbf{L}, \mathbf{S}, \mathbf{R}, \mathbf{L}^0$ and $\mathbf{R}^0$. Hence, each of these quasi-varieties is generated by the direct product of two dualisable algebras from two different varieties. However, it is not always true that the direct product of dualisable algebras is dualisable as shown in Example 2.4.5. We modify Example 2.4.5 in the last section of this chapter by showing the product of a dualisable groupoid with constant (and term equivalent to a Boolean algebra) with a dualisable 2-element right-zero semigroup with constant is non-dualisable.

A quasi-variety admits a natural duality if it is generated by a finite algebra. Every quasi-variety of normal bands is generated by some finite normal band. Saramago [43] and Davey and Willard [17] (Theorem 2.4.4) showed that if two algebras generate the same quasi-variety and one of them is dualisable then the other is dualisable. In all proofs, we consider an algebra **D** that generates the specified quasi-variety of normal bands, and show that **D** is dualisable. Moreover, in each case the alter ego of the algebra **D** is of finite type, hence it is sufficient to show that (IC) holds. Our dualities will be built upon five existing dualities namely, the dualities for the quasi-varieties generated by the 2-element semilattice **S**, the 2-element left-zero semigroup **L**, the 2-element right-zero semigroup **R**, the 4-element rectangular band **RB** and the 3-element left normal band $\mathbf{L}^0$. We will conclude with three lemmas that will be used in the rest of this section. The first is general and the second and third refer to the 2-element semilattice and the 2-element left- or right-zero semigroup, respectively.

Definition 3.2.1. A family of maps $\{\varphi_i : X \to Y_i \mid i \in I\}$ is *separating* if for all $x \neq y$ in X, there is $i \in I$ with $\varphi_i(x) \neq \varphi_i(y)$.

Lemma 3.2.2. *Let* **D** *be a finite algebra, let* **M** *and* **N** *be subalgebras of* **D** *and* $g_M : D \to M$, $g_N : D \to N$ *be homomorphisms onto* **M** *and* **N**, *respectively with* $\{g_M, g_N\}$ *separating. Assume* $\underset{\sim}{\mathbf{D}}$ *is an alter ego of* **D** *that includes the unary homomorphisms* g_M, g_N. *Consider* $\mathbf{X} \leqslant \underset{\sim}{\mathbf{D}}^n$, *for some* $n \in \mathbb{N}$ *and assume that* $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ *is a morphism such that there is a term* $t(x_1, \ldots, x_n)$ *with*

$$(\forall x \in X \cap M^n)\ \varphi(x) = t(x) \quad \textit{and} \quad (\forall x \in X \cap N^n)\ \varphi(x) = t(x).$$

Then $\varphi(x) = t(x)$, *for every* $x \in X$.

Proof. Let $x \in X$. As $g_M(x) \in X \cap M^n$, we have $\varphi(g_M(x)) = t(g_M(x))$ which implies that $g_M(\varphi(x)) = g_M(t(x))$. Similarly, $g_N(x) \in X \cap N^n$ implies $\varphi(g_N(x)) = t(g_N(x))$. Then we have $g_N(\varphi(x)) = g_N(t(x))$. Since g_M, g_N are separating homomorphisms, therefore $\varphi(x) = t(x)$, for every $x \in X$. □

The following two basic lemmas will essentially establish (IC) for **S** and **L** (whence **R**); we need the precise details for our proofs later.

Lemma 3.2.3. *Let* $\underset{\sim}{\mathbf{S}} = \langle \{a, b\}; *, a, b\rangle$ *be the* 2*-element bounded semilattice with* $*$ *the semilattice operation and* $a * b = b$. *For* $\mathbf{X} \leqslant \underset{\sim}{\mathbf{S}}^n$, *let* $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{S}}$ *be a morphism. Let* $\widehat{a}$ *be the* $*$*-product of all elements of the set* $\{x \in X \mid \varphi(x) = a\}$ *which contains* $\underline{a}$ *and* $I = \{i \le n \mid (\widehat{a})_i = a\}$. *Then the set* $I \neq \varnothing$ *and for all* $x \in X$, *we have* $\varphi(x) = x_{i_1} * \cdots * x_{i_m}$ *for* $\{i_1, \ldots, i_m\} = I$.

Proof. We know $\underline{b} \neq \widehat{a}$ since $\varphi(\underline{b}) = b$ and $\varphi(\underline{a}) = a$, and it follows that the set I is nonempty. We will prove that for $x \in X$, we have $\varphi(x) = a$ if and only if $x_i = a$ for all $i \in I$. By the definition of $\widehat{a}$ and I, it follows that for $x \in X$ with $\varphi(x) = a$ we have $x_i = a$ for all $i \in I$. Conversely, if for all $i \in I$ we have $x_i = a$ then $x * \widehat{a} = \widehat{a}$. It follows that $a = \varphi(\widehat{a}) = \varphi(x) * \varphi(\widehat{a}) = \varphi(x) * a = \varphi(x)$.

Equivalently, for $x \in X$, we have $\varphi(x) = b$ if and only if there exists $j \in I$ with $x_j = b$. Hence for all $x \in X$ we have $\varphi(x) = x_{i_1} * \cdots * x_{i_m}$ for $\{i_1, \ldots, i_m\} = I$. □

Lemma 3.2.4. *Let* $\underset{\sim}{\mathbf{L}} = \langle \{a, b\}; *, \vee, \wedge, '\rangle$ *be an algebra where* $\langle \{a, b\}; \vee, \wedge, '\rangle$ *is a Boolean algebra with* $a < b$ *and* $\langle \{a, b\}; *\rangle$ *is a* 2*-element left-zero or right-zero semigroup. For* $\mathbf{X} \leqslant \underset{\sim}{\mathbf{L}}^n$, *let* $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{L}}$ *be a homomorphism. Let* $J = \{j \le n \mid (\forall x \in X)\ \varphi(x) = x_j\}$ *and define* $\check{a} = \bigvee \varphi^{-1}(a)$. *Then the set* $J \neq \varnothing$ *and we have* $(\check{a})_j = a$ *if and only if* $j \in J$.

Proof. Let $X \neq \varnothing$. Since $\varphi(\underline{a}) = a$, it follows that $\varphi^{-1}(a) \neq \varnothing$. Hence, $\check{a}$ is well defined. Suppose that $(\check{a})_j = a$. We shall prove that $\varphi(x) = x_j$ for every $x \in X$. Either $\varphi(x) = a$ or $\varphi(x) = b$, suppose the first case. Hence $x \in \varphi^{-1}(a)$ and $\check{a} \vee x = \check{a}$. However, $\check{a}_j \vee x_j = \check{a}_j = a$. Therefore, $x_j = a = \varphi(x)$. In the second case, if $\varphi(x) = b$, then $\varphi(x') = a$. Applying the above argument proves that $(x')_j = a$, whence $x_j = b = \varphi(x)$.

To prove that $J \neq \varnothing$, it is enough to show that $\breve{a} \neq \underline{b}$. Suppose by way of contradiction that $\breve{a} = \underline{b}$, then $(\breve{a})' = \underline{a}$. However, $\breve{a} = \breve{a} \vee (\breve{a})'$ which implies that $\varphi(\breve{a}) = \varphi(\breve{a}) \vee \varphi((\breve{a})') = a \vee b = b$, a contradiction. □

We introduce the following notation which will be required in the proofs.

Notation 3.2.5. Let $M = \{x, y\}$ and $N = \{x, y, z\}$. Define the binary operations $\vee_{x,y} \colon M^2 \to M$, $\wedge_{x,y} \colon M^2 \to M$, $\vee_{x,y,z} \colon N^2 \to N$ and $\wedge_{x,y,z} \colon N^2 \to N$ as follows:

$\vee_{x,y}$	x	y
x	x	y
y	y	y

,

$\wedge_{x,y}$	x	y
x	x	x
y	x	y

,

$\vee_{x,y,z}$	x	y	z
x	x	y	z
y	y	y	z
z	z	z	z

,

$\wedge_{x,y,z}$	x	y	z
x	x	x	x
y	x	y	y
z	x	y	z

.

Note that $\vee_{x,y}$ ($\wedge_{x,y}$) is the join (meet) in the chain $x < y$ and $\vee_{x,y,z}$ ($\wedge_{x,y,z}$) is the join (meet) in the chain $x < y < z$.

3.2.1 $\mathcal{L} \vee \mathcal{S}$ duality

We will show that the quasi-variety generated by the product of the left-zero semigroup **L** with the 2-element semilattice **S** is dualisable by finding an alter ego satisfying (IC). By symmetry, the quasi-variety generated by the product of the 2-element semilattice **S** with the 2-element right-zero semigroup **R** is also dualisable. Consider the semigroup **D** on $\{(i, s) \mid i, s \in \{0, 1\}\}$ with the multiplication $*$ given by

$$(\forall (i_1, s_1), (i_2, s_2) \in D) \quad (i_1, s_1) * (i_2, s_2) = (i_1, s_1 \cdot s_2).$$

It is easy to check that **D** is isomorphic to $\mathbf{L} \times \mathbf{S}$. Note that for $k \in \{0, 1\}$,

$$L_k = \{(i, k) \mid i \in \{0, 1\}\}, \; S_k = \{(k, s) \mid s \in \{0, 1\}\}$$

form 2-element left-zero subsemigroups and 2-element subsemilattices, respectively, under $*$. Let $u_k\colon \mathbf{D} \to \mathbf{D}$, $'\colon \mathbf{D} \to \mathbf{D}$ and $w_0\colon \mathbf{D} \to \mathbf{D}$ be endomorphisms defined as follows:

$$u_k((i,s)) = (i,k),\ w_0((i,s)) = (0,s), \quad (i,s)' = (i',s)$$

where $'$ is the complement operation on $\{0,1\}$ such that $0' = 1$ and $1' = 0$.

We will not notationally distinguish between an endomorphism and its restriction to a subset of its domain. To make the proof notationally easier to read, let

$$a = (0,1),\ b = (1,1),\ c = (0,0) \text{ and } d = (1,0).$$

Let $\vee_{a,b}$, $\wedge_{a,b}$ be binary operations on L_1 as defined in Notation 3.2.5. Similarly, define the binary operations $\vee_{c,d}$, $\wedge_{c,d}$ on L_0.

By Lemma 3.2.4 and Theorem 2.4.3, $\langle L_1; *, \vee_{a,b}, \wedge_{a,b}, ', a, b, \mathfrak{T}\rangle$ and $\langle L_0; *, \vee_{c,d}, \wedge_{c,d}, ', c, d, \mathfrak{T}\rangle$ dualise $\langle L_1; *\rangle$ and $\langle L_0; *\rangle$, respectively. By Lemma 3.2.3 and Theorem 2.4.3, $\langle S_0; *, a, c, \mathfrak{T}\rangle$ and $\langle S_1; *, b, d, \mathfrak{T}\rangle$ dualise $\langle S_0; *\rangle$ and $\langle S_1; *\rangle$, respectively. Observe that $\vee_{c,d} = u_0 \circ \vee_{a,b} \circ (u_1 \times u_1)$ and similarly $\wedge_{c,d} = u_0 \circ \wedge_{a,b} \circ (u_1 \times u_1)$. Finally, the set $\rhd = \{b, c, d\}$ forms a subsemigroup of $\mathbf{D}$. Let

$$G^D = \{*, u_1, u_0, w_0, ', a, b, c, d\}$$

and

$$H^D = \{\vee_{a,b}, \wedge_{a,b}\}.$$

Theorem 3.2.6. *The alter ego*

$$\underset{\sim}{\mathbf{D}} = \langle \{a,b,c,d\}; G^D, H^D, \rhd, \mathfrak{T}\rangle$$

dualises $\mathbf{D}$ *and hence* $\mathcal{L} \vee \mathcal{S}$ *has a natural duality.*

Proof. Since $\underset{\sim}{\mathbf{D}}$ is of finite type, by the IC Duality Theorem 2.4.3, it suffices to prove that $\underset{\sim}{\mathbf{D}}$ satisfies (IC). Let $n \in \mathbb{N}$ and $\mathbf{X} \leqslant \underset{\sim}{\mathbf{D}}^n$. Let $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ be a morphism. We will apply Lemma 3.2.2 on $\mathbf{D}$ with subalgebra $\mathbf{M}$ chosen to be $\mathbf{L}_1$ and subalgebra $\mathbf{N}$ chosen to be $\mathbf{S}_0$.

Now we consider first $\mathbf{L}_1$. As every term function of $\mathbf{L}_i$ is a projection, for all $x \in X \cap L_i^n$, we have $\varphi(x) = x_j$, for some $j \in \{1, \dots, n\}$. Let

$$I_{L_1} = \{i \leq n \mid (\forall x \in X \cap L_1^n)\ \varphi(x) = x_i\}.$$

Define $\breve{a} = \bigvee_{a,b} u_1(\varphi^{-1}(a))$. Hence we have $\breve{a} \in X \cap L_1^n$ and $\varphi(\breve{a}) = a$. Applying Lemma 3.2.4 on $\langle L_1; *, \vee_{a,b}, \wedge_{a,b}, ' \rangle$, we have $(\breve{a})_i = a$, for $i \in I_{L_1}$, and $(\breve{a})_k = b$, for $k \notin I_{L_1}$.

We now consider $\mathbf{S}_0$. Define $\widehat{a}$ to be the $*$-product of all elements of the set $w_0(\varphi^{-1}(a))$ and let

$$I_{S_0} = \{i \in \{1, \dots, n\} \mid (\widehat{a})_i = a\} = \{i_1, \dots, i_m\}.$$

Applying Lemma 3.2.3 on $\langle S_0; *, a, c \rangle$, for every $x \in X \cap S_0^n$ we have $\varphi(x) = x_{i_1} * \cdots * x_{i_m}$, where $\{i_1, \dots, i_m\} = I_{S_0}$, and $(\widehat{a})_i = a$, for $i \in I_{S_0}$, and $(\widehat{a})_l = c$, for $l \notin I_{S_0}$.

Now we claim that $I_{S_0} \cap I_{L_1} \neq \varnothing$. Suppose to the contrary that I_{S_0} and I_{L_1} are disjoint. Without loss of generality, we may assume that $I_{S_0} = \{1, \dots, m\}$ and $I_{L_1} = \{m+1, \dots, m + |I_{L_1}|\}$. The following table will give a contradiction to the assumption $I_{S_0} \cap I_{L_1} = \varnothing$.

x	1	$\dots$	m	$m+1$	$\dots$	$\lvert I_{L_1}\rvert + m$	$\lvert I_{L_1}\rvert + m + 1$	$\dots$	n	$\varphi(x)$
$\breve{a}$	b	$\dots$	b	a	$\dots$	a	b	$\dots$	b	a
$\widehat{a}$	a	$\dots$	a	c	$\dots$	c	c	$\dots$	c	a
$\breve{a} * \widehat{a}$	b	$\dots$	b	c	$\dots$	c	d	$\dots$	d	a

The last line shows that φ does not preserve the relation $\vartriangleright$, a contradiction. Hence $I_{S_0} \cap I_{L_1} \neq \varnothing$. We are now in a position to apply Lemma 3.2.2. Let $j \in I_{S_0} \cap I_{L_1}$ and $t\colon \mathbf{D}^n \to \mathbf{D}$ be a term function given by

$$t(x_1, \dots, x_n) = x_j * x_{i_1} * \cdots * x_{i_m}.$$

For all $x \in X \cap S_0^n$, we have $\varphi(x) = x_{i_1} * \cdots * x_{i_m}$, which equals $t(x_1, \dots, x_n)$ on $\mathbf{S}_0$, and for all $x \in X \cap L_1^n$, we have $\varphi(x) = x_j$, for some $j \in I_{S_0} \cap I_{L_1}$, which equals $t(x_1, \dots, x_n)$ on $\mathbf{L}_1$. Since u_1 and w_1 are separating retracts onto $\mathbf{L}_1$ and $\mathbf{S}_0$, respectively, Lemma 3.2.2 shows that $\varphi(x) = t(x)$, for all $x \in X$. □

3.2.2 $\mathcal{RB} \vee \mathcal{S}$ duality

Let $\mathbf{M}$ be a finite rectangular band. Then by Davey and Knox [14] $\mathbf{M}$ is dualisable by some alter ego $\underset{\sim}{\mathbf{M}}$ of finite type. Let $\mathbf{S}$ be a 2-element semilattice. Then by Hofmann, Mislove and Stralka [27], $\mathbf{S}$ is dualisable by some alter ego $\underset{\sim}{\mathbf{S}}$ of finite type. Consider the semigroup $\mathbf{D}$ on $\{(i, s, j) \mid i, s, j \in \{0, 1\}\}$ with the multiplication $*$ given by

$$(\forall (i_1, s_1, j_1), (i_2, s_2, j_2) \in D) \quad (i_1, s_1, j_1) * (i_2, s_2, j_2) = (i_1, s_1 \cdot s_2, j_2).$$

It is clear that $\mathbf{D} \cong \mathbf{L} \times \mathbf{S} \times \mathbf{R}$ where $\mathbf{L}$ and $\mathbf{R}$ are left-zero semigroup and right-zero semigroup, respectively. Note that $M_1 = \{(i, 1, j) \mid i, j \in \{0, 1\}\}$, $M_0 = \{(i, 0, j) \mid i, j \in \{0, 1\}\}$ form rectangular bands under $*$. Notice also that for fixed coordinates i, j, the sets $S_{ij} = \{(i, s, j) \mid s \in \{0, 1\}\}$ form 2-element subsemilattices under $*$. Similarly, we define the sets

$$R_{is} = \{(i, s, j) \mid j \in \{0, 1\}\} \quad \text{and} \quad L_{sj} = \{(i, s, j) \mid i \in \{0, 1\}\}$$

which form right-zero subsemigroups and left-zero subsemigroups under $*$, respectively. Observe that $\mathbf{D}$ generates the quasi-variety join $\mathcal{RB} \vee \mathcal{S}$. (See Figure 3.2.)

For $k \in \{0,1\}$, let $\sigma_k : D \to D$, $\rho_k : D \to D$, $\lambda_k : D \to D$, $\kappa_l : D \to D$ and $\kappa_r : D \to D$ be endomorphisms of $\mathbf{D}$ defined as follows:

$$\sigma_k((i,s,j)) = (i,k,j); \quad \rho_k((i,s,j)) = (i,s,k); \quad \lambda_k((i,s,j)) = (k,s,j);$$
$$\kappa_l((i,s,j)) = (i^{'},s,j); \quad \kappa_r((i,s,j)) = (i,s,j^{'});$$

where $'$ is the complement operation on the set $\{0,1\}$ such that $0' = 1$, $1' = 0$. To make the proof notationally easier, let

$$a = (0,1,0),\ b = (0,1,1),\ c = (1,1,0),\ d = (1,1,1),$$
$$e = (0,0,0),\ f = (0,0,1),\ g = (1,0,0),\ h = (1,0,1)$$

be the elements of the set D. Let $\vee_{a,b}$, $\wedge_{a,b}$ be binary operations on $R_{01} = \{a,b\}$ as defined in Notation 3.2.5. Similarly, we define the remaining binary operations $\vee_Z, \wedge_Z$ on $Z \in \{L_{sj}, R_{is}\}$. Observe that

$$\vee_{c,d} = \kappa_l \circ \vee_{a,b} \circ (\kappa_l \times \kappa_l),$$
$$\vee_{e,f} = \sigma_0 \circ \vee_{a,b} \circ (\sigma_1 \times \sigma_1),$$
$$\vee_{g,h} = \sigma_0 \circ \kappa_l \circ \vee_{a,b} \circ ((\kappa_l \circ \sigma_1) \times (\kappa_l \circ \sigma_1)),$$

and similarly, the remaining partial operations $\wedge_Z$ and $\vee_Z$ can be expressed in term of $\wedge_{a,b}$, $\vee_{a,c}$, $\wedge_{a,c}$ and some endomorphisms. Let

$$G^D = \{*, \kappa_r, \kappa_l\} \cup D \cup \{\lambda_k, \sigma_k, \rho_k \mid k \in \{0,1\}\}$$

and let

$$H^D = \{\wedge_{a,b}, \wedge_{a,c}, \vee_{a,b}, \vee_{a,c}\}.$$

Finally, the sets $\rhd_1 = \{e, c, g\}$ and $\rhd_2 = \{e, b, f\}$ form subsemigroups of $\mathbf{D}$. Notice that by Lemma 3.2.3,

$$\underset{\sim}{\mathbf{S}}_{00} = \langle S_{00}; *, a, e, \mathcal{T}\rangle$$

dualises $\mathbf{S}_{00}$. We will show in the proof of Theorem 3.2.7 that

$$\underset{\sim}{\mathbf{M}}_1 = \langle M_1; *, \vee_{a,b}, \vee_{a,c}, \wedge_{a,b}, \wedge_{a,c}, \kappa_r, \kappa_l, a, b, c, d, \mathcal{T}\rangle$$

dualises the rectangular band $\mathbf{M}_1$.

Theorem 3.2.7. *The alter ego*

$$\underset{\sim}{\mathbf{D}} = \langle D; G^D, H^D, \{\rhd_1, \rhd_2\}, \mathcal{T}\rangle$$

dualises $\mathbf{D}$ *and hence* $\mathcal{RB} \vee \mathcal{S}$ *has a natural duality.*

Proof. We may apply the IC Duality Theorem 2.4.3 since the alter ego $\underset{\sim}{\mathbf{D}}$ is of finite type. Let $n \in \mathbb{N}$ and $\mathbf{X} \leqslant \underset{\sim}{\mathbf{D}}^n$. Let $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ be a morphism. We will apply Lemma 3.2.2 on $\mathbf{D}$ with $\mathbf{M}$ chosen to be the subsemigroup $\mathbf{M}_1$ and $\mathbf{N}$ chosen to be the subsemigroup $\mathbf{S}_{00}$.

First, we consider the rectangular band $\mathbf{M}_1$. It is easy to check that $\mathbf{M}_1 \cong \mathbf{L}_{10} \times \mathbf{R}_{01}$. Let

$$I_{L_{10}} := \{k \in \{1, \dots, n\} \mid (\forall x \in X \cap L_{10}^n) \quad \varphi(x) = x_k\}.$$

Now define $(\breve{a})^L = \bigvee_{a,c} \rho_0(\sigma_1(\varphi^{-1}(a)))$; then $(\breve{a})^L \in X \cap L_{10}^n$ and $\varphi((\breve{a})^L) = a$. Applying Lemma 3.2.4 on $\langle L_{10}; *, \vee_{a,c}, \wedge_{a,c}, \kappa_l\rangle$, we have $(\breve{a})^L_k = a$, for $k \in I_{L_{10}}$, and $(\breve{a})^L_l = c$, for $l \notin I_{L_{10}}$. Let

$$I_{R_{01}} := \{j \in \{1, \dots, n\} \mid (\forall x \in X \cap R_{01}^n) \quad \varphi(x) = x_j\}.$$

Define $(\check{a})^R = \bigvee_{a,b} \lambda_0(\sigma_1(\varphi^{-1}(a)))$; then $(\check{a})^R \in X \cap R_{01}^n$ and $\varphi((\check{a})^R) = a$. Then by applying Lemma 3.2.4 on $\langle R_{01}; *, \vee_{a,b}, \wedge_{a,b}, \kappa_r \rangle$, we have $(\check{a})_j^R = a$, for $j \in I_{R_{01}}$, and $(\check{a})_q^R = b$, for $q \notin I_{R_{01}}$.

We now consider $\mathbf{S}_{00}$. Define $\widehat{a}$ to be $*$-product of all elements of the set $\lambda_0 \circ \rho_0(\varphi^{-1}(a))$ and let

$$I_{S_{00}} = \{i \leq n \mid (\widehat{a})_i = a\} = \{i_1, \dots, i_m\}.$$

Applying Lemma 3.2.3 on $\langle S_{00}; *, a, e \rangle$, for every $x \in X \cap S_{00}^n$ we have $\varphi(x) = x_{i_1} * \dots * x_{i_m}$, for $\{i_1, \dots, i_m\} = I_{S_{00}}$. Moreover, we have $(\widehat{a})_i = a$, for $i \in I_{S_{00}}$ and $(\widehat{a})_s = e$, for $s \notin I_{S_{00}}$.

We claim that $I_{S_{00}} \cap I_{L_{10}} \neq \varnothing$ and $I_{S_{00}} \cap I_{R_{01}} \neq \varnothing$. Suppose by way of contradiction that $I_{S_{00}} \cap I_{L_{10}} = \varnothing$. Without loss of generality, we may assume that $I_{S_{00}} = \{1, \dots, m\}$, $I_{L_{10}} = \{m+1, \dots, |I_{L_{10}}| + m\}$. The table below will give us a contradiction to the assumption that $I_{S_{00}} \cap I_{L_{10}} = \varnothing$.

x	$1 \quad \dots \quad m$	$m+1 \quad \dots \quad m+\lvert I_{L_{10}}\rvert$	$m+\lvert I_{L_{10}}\rvert+1 \quad \dots \quad n$	$\varphi(x)$
$(\check{a})^L$	$c \quad \dots \quad c$	$a \quad \dots \quad a$	$c \quad \dots \quad c$	a
$\widehat{a}$	$a \quad \dots \quad a$	$e \quad \dots \quad e$	$e \quad \dots \quad e$	a
$(\check{a})^L * \widehat{a}$	$c \quad \dots \quad c$	$e \quad \dots \quad e$	$g \quad \dots \quad g$	a

The last line shows that φ does not preserve the relation $\rhd_1$, hence $I_{S_{00}} \cap I_{L_{10}} \neq \varnothing$. By symmetry, we have $I_{S_{00}} \cap I_{R_{01}} \neq \varnothing$.

We are now in a position to apply Lemma 3.2.2. Let $i_L \in I_{S_{00}} \cap I_{L_{10}}$ and $i_R \in I_{S_{00}} \cap I_{R_{01}}$. Let $t\colon \mathbf{D}^n \to \mathbf{D}$ be the term function given by

$$t(x_1, \dots, x_n) = x_{i_L} * x_{i_1} * \dots * x_{i_m} * x_{i_R}.$$

Then, for all $x \in X \cap M_1^n$, we have $\varphi(x) = x_{i_L} * x_{i_R}$ which equals $t(x_1, \dots, x_n)$ on $\mathbf{M}_1$, and for all $x \in X \cap S_{00}^n$, we have $\varphi(x) = x_{i_1} * \dots * x_{i_m}$, which equals

$t(x_1, \ldots, x_n)$ on $\mathbf{S}_{00}$. Since σ_1 and $\lambda_0 \circ \rho_0$ are separating retracts onto $\mathbf{M}_1$ and $\mathbf{S}_{00}$, respectively, Lemma 3.2.2 shows that $\varphi(x) = t(x)$, for all $x \in X$. □

3.2.3 $\mathcal{L}^0 \vee \mathcal{R}$ duality

We will show that the quasi-variety generated by the product of the left normal band $\mathbf{L}^0$ and the right-zero semigroup $\mathbf{R}$ is dualisable (and by symmetry we conclude that the quasi-variety generated by the product of the right normal band and the left-zero semigroup is dualisable) by showing that it has an alter ego that satisfies (IC). Let $\mathbf{D}$ be the semigroup on

$$\{(i, j) \mid i \in \{0, 1, 2\} \text{ and } j \in \{0, 1\}\}$$

with multiplication $*$ defined by:

$$(\forall (i_1, j_1), (i_2, j_2) \in D) \quad (i_1, j_1) * (i_2, j_2) = \begin{cases} (i_1, j_2) & \text{if } i_1 \neq 0 \text{ and } i_2 \neq 0, \\ (0, j_2) & \text{otherwise.} \end{cases}$$

Note that for $l \in \{0, 1, 2\}$, $k \in \{0, 1\}$, the sets $R_l = \{(l, j) \mid j \in \{0, 1\}\}$ form 2-element right-zero subsemigroups and $M_k = \{(i, k) \mid i \in \{0, 1, 2\}\}$ form left normal idempotent subsemigroups under $*$. For $k \in \{0, 1\}$, the sets $L_k = \{(i, k) \mid i \in \{1, 2\}\}$ form 2-element left-zero subsemigroups and

$$S_1 = \{(i, 1) \mid i \in \{0, 1\}\}, \; S_2 = \{(i, 1) \mid i \in \{0, 2\}\},$$
$$S_3 = \{(i, 0) \mid i \in \{0, 1\}\}, \; S_4 = \{(i, 0) \mid i \in \{0, 2\}\}$$

form 2-element subsemilattices under $*$. Observe that $\mathbf{D}$ generates the quasi-variety $\mathcal{L}^0 \vee \mathcal{R}$. Let $u_k \colon \mathbf{D} \to \mathbf{D}$, $w_l \colon \mathbf{D} \to \mathbf{D}$, $' \colon \mathbf{D} \to \mathbf{D}$ and

$\sharp\colon \mathbf{D} \to \mathbf{D}$ be endomorphisms defined as follows:

$$u_k((i,j)) = (i,k),\ w_l((i,j)) = (l,j),\ (i,j)' = (i,j^{'})$$

and

$$\sharp((i,j)) = \begin{cases} (0,j) & \text{if } i = 0, \\ (1,j) & \text{if } i = 2, \\ (2,j) & \text{if } i = 1, \end{cases}$$

where $'$ is the complement on the set $\{0,1\}$. To make it notationally easier for the reader, we let

$$0 = (0,1),\ a = (1,1),\ b = (2,1),\ 0^{'} = (0,0),\ c = (1,0) \text{ and } d = (2,0).$$

Let $\vee_{a,c}, \wedge_{a,c}$ be binary operations on $R_1 = \{a,c\}$ as defined in Notation 3.2.5. Similarly, we define the following operations $\vee_{b,d}, \wedge_{b,d}, \vee_{0,0'}, \wedge_{0,0'}$, $\vee_{a,b,0}$, $\wedge_{0,a,b}$, $\vee_{c,d,0'}$ and $\wedge_{0',c,d}$. Observe that $u_0 = {'} \circ u_1$, $w_2 = \sharp \circ w_1$, $\vee_{b,d} = \sharp \circ \vee_{a,c} \circ (\sharp \times \sharp)$, $\wedge_{b,d} = \sharp \circ \wedge_{a,c} \circ (\sharp \times \sharp)$, $\vee_{c,d,0'} = u_0 \circ \vee_{a,b,0} \circ (u_1 \times u_1)$ and $\wedge_{0',c,d} = u_0 \circ \wedge_{0,a,b} \circ (u_1 \times u_1)$. We define $\vee_{a,b}$ to be the binary operation $\vee_{a,b,0}$ restricted to the set L_1, similarly for $\wedge_{a,b}$, $\vee_{c,d}$ and $\wedge_{c,d}$. Let

$$G^D = \{*, ', \sharp, 0, a, b, 0^{'}, c, d\} \cup \{u_1, w_1, w_0\}$$

and let

$$H^D = \{\vee_{a,c}, \wedge_{a,c}, \vee_{0,0'}, \wedge_{0,0'}, \vee_{a,b}, \wedge_{a,b}, \vee_{a,b,0}, \wedge_{0,a,b}\}.$$

Finally, the set $\triangleright = \{0, 0^{'}, c, d\}$ forms a subsemigroup of $\mathbf{D}$. Notice that by using Lemma 3.2.4, the alter ego $\underset{\sim}{\mathbf{L}}_1 = \langle L_1; *, \vee_{a,b}, \wedge_{a,b}, \sharp, a, b, \mathcal{T}\rangle$ dualises $\mathbf{L}_1$ and $\underset{\sim}{\mathbf{R}}_1 = \langle R_1; *, \vee_{a,c}, \wedge_{a,c}, ', a, c, \mathcal{T}\rangle$ dualises $\mathbf{R}_1$. We will show in the proof of Theorem 3.2.8 that

$$\underset{\sim}{\mathbf{M}}_1 = \langle M_1; *, \vee_{a,b,0}, \wedge_{0,a,b}, \sharp, a, b, 0, \mathfrak{T}\rangle$$

dualises the left normal band $\mathbf{M}_1$.

Theorem 3.2.8. *The alter ego*

$$\underset{\sim}{\mathbf{D}} = \langle \{a, b, c, d, 0, 0'\}; G^D, H^D, \{\triangleright\}, \mathfrak{T}\rangle$$

dualises $\mathbf{D}$ *and hence* $\mathcal{L}^0 \vee \mathcal{R}$ *has a natural duality.*

Proof. We apply the IC Duality Theorem 2.4.3. Let $n \in \mathbb{N}$ and $\mathbf{X} \leqslant \underset{\sim}{\mathbf{D}}^n$. Consider a morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$. We will apply Lemma 3.2.2 on $\mathbf{D}$ with the subalgebra $\mathbf{M}$ chosen to be the subsemigroup $\mathbf{M}_1$ and the subalgebra $\mathbf{N}$ to be the subsemigroup $\mathbf{R}_1$.

First, we consider $\mathbf{M}_1$. Let $A = \{x \in X \cap M_1^n \mid \varphi(x) \neq 0\}$ and define $\widehat{x}$ to be the $*$-product of all elements of A relative to some fixed ordering, with the constant $\underline{a}$ first. Let $I_A = \{i \leq n \mid \widehat{x}_i \neq 0\}$. Hence by definition of $\widehat{x}$ and I_A, we have for all $x \in A$, $x_i \neq 0$ for all $i \in I_A$. Conversely, let $x \in X \cap M_1^n$ with $x_i \neq 0$, for all $i \in I_A$. Then $\widehat{x} * x = \widehat{x}$, showing $\varphi(x) \neq 0$ and $x \in A$. Therefore, for all $x \in X \cap M_1^n$, we have $\varphi(x) \neq 0$ if and only if $x_i \neq 0$ for all $i \in I_A$. Equivalently, for all $x \in X \cap M_1^n$, we have $\varphi(x) = 0$ if and only if there exists $i \in I_A$ such that $x_i = 0$. Let $\pi_{I_A}\colon D^n \to D^{I_A}$ be the restriction to I_A. It is clear that $\pi_{I_A}(A) \subseteq L_1^{I_A}$. As $\vee_{a,b,0}, \wedge_{0,a,b}, \sharp$ are total on $\mathbf{M}_1$, $\mathbf{L}_1$ is subuniverse of $\langle M_1; *, \vee_{a,b,0}, \wedge_{0,a,b}, \sharp, a, b, 0\rangle$ and $\underline{a}, \underline{b} \in \pi_{I_A}(A)$, then $\pi_{I_A}(A) \leqslant \underset{\sim}{\mathbf{L}}_1^{I_A}$.

Let $\psi\colon \pi_{I_A}(A) \to L_1$ be given for all $z \in \pi_{I_A}(A)$ by $\psi(z) = \varphi(x)$, where $x \in A$ and $\pi_{I_A}(x) = z$. Then ψ is well defined if and only if $\ker(\pi_{I_A}) \subseteq \ker(\varphi)$. We will argue that ψ is well defined, that is, if $x, y \in A$ with $\pi_{I_A}(x) = \pi_{I_A}(y)$ we will show that $\varphi(x) = \varphi(y)$. Since $x*\widehat{x} = y*\widehat{x}$ and φ

preserves $*$ then $\varphi(x) = \varphi(x * \widehat{x}) = \varphi(y * \widehat{x}) = \varphi(y)$. Hence $\ker(\pi_{I_A}) \subseteq \ker(\varphi)$, that is, ψ is a unique morphism such that $\psi \circ \pi_{I_A} = \varphi$. Since $\underset{\sim}{\mathbf{L}}_1$ dualises $\mathbf{L}_1$ with (IC), therefore the morphism ψ extends to the term $t = x_i$ for some $i \in I_A$. Define the term function s to be $s = x_i * x_{i_1} \cdots * x_{i_m}$ where $I_A = \{i_1, \ldots, i_m\}$ and $m \leq n$. It easy to see that t is equivalent to s on $\mathbf{L}_1$. If $x \in (X \cap M_1) \backslash A$ then $\varphi(x) = 0$ and there exists $l \in I_A$ such that $x_l = 0$. Hence $s(x) = 0 = \varphi(x)$. Thus for all $x \in X \cap M_1$, we have $\varphi(x) = s(x)$.

We now consider subsemigroup $\mathbf{R}_1$. Let

$$I_{R_1} = \{j \in \{1, \ldots, n\} \mid (\forall x \in X \cap R_1^n) \quad \varphi(x) = x_j\}$$

and define $\check{a} = \bigvee_{a,c} w_1(\varphi^{-1}(a))$. Applying Lemma 3.2.4 on

$$\langle X \cap R_1^n; *, \vee_{a,c}, \wedge_{a,c}, ' \rangle,$$

we have $I_{R_1} \neq \varnothing$. Moreover, we have $(\check{a})_j = a$, for $j \in I_{R_1}$ and $(\check{a})_k = c$, for $k \notin I_{R_1}$.

We will argue that $I_{R_1} \cap I_A \neq \varnothing$. Suppose by way of contradiction that $I_{R_1} \cap I_A = \varnothing$. Without loss of generality we may assume that $I_{R_1} = \{1, \ldots, |I_{R_1}|\}$ and $I_A = \{|I_{R_1}| + 1, \ldots, |I_{R_1}| + m\}$. Then the following table will give us a contradiction to the assumption $I_{R_1} \cap I_A = \varnothing$.

x	1	$\ldots$	$\lvert I_{R_1}\rvert$	$\lvert I_{R_1}\rvert + 1$	$\ldots$	$\lvert I_{R_1}\rvert + m$	$\lvert I_{R_1}\rvert + m + 1$	$\ldots$	n	$\varphi(x)$
$\widehat{x}$	0	$\ldots$	0	a, b	$\ldots$	a, b	0	$\ldots$	0	a, b
$\check{a}$	a	$\ldots$	a	c	$\ldots$	c	c	$\ldots$	c	a
$\widehat{x} * \check{a}$	0	$\ldots$	0	c, d	$\ldots$	c, d	$0'$	$\ldots$	$0'$	a, b

The last line shows that φ does not preserve the relation $\triangleright$, a contradiction. We are now in a position to apply Lemma 3.2.2. Let $j \in I_A \cap I_{R_1}$ and

$t\colon \mathbf{D}^n \to \mathbf{D}$ be a term function given by

$$t(x_1, \dots, x_n) = s(x) * x_j.$$

For all $x \in X \cap M_1^n$ we have $\varphi(x) = s(x)$ which equals $t(x_1, \dots, x_n)$ on $\mathbf{M}_1$, and for all $x \in X \cap R_1^n$ we have $\varphi(x) = x_j$, for some $j \in I_A \cap I_{R_1}$, which equals $t(x_1, \dots, x_n)$ on $\mathbf{R}_1$. Since u_1 and w_1 are separating retracts onto $\mathbf{M}_1$ and $\mathbf{R}_1$, respectively, Lemma 3.2.2 shows that $\varphi(x) = t(x)$, for all $x \in X$. □

We conclude this section with two corollaries to the main results in the chapter and the definition of finite degree.

Corollary 3.2.9. *For a finite band* $\mathbf{M}$*, the following are equivalent:*

(1) $\mathbf{M}$ *is dualisable;*

(2) $\mathbf{M}$ *is a normal band;*

(3) $\mathbf{M}$ *is not inherently non-dualisable;*

(4) $\mathbb{HSP}(\mathbf{M})$ *has a finite residual bound.*

Definition 3.2.10. An algebra $\mathbf{A}$ has *finite degree* if there is a finite set R of algebraic relations on A, such that $\mathbf{A}$ has (CLO) with respect $\langle A; R\rangle$. (Algebras with finite degree are also known as *finitely related.*)

There is a deep unpublished result establishes the finite degree of finite groups [1]. It is known that every semilattice [7] and rectangular band has a finite degree [[14], [6]]. At this point it has been proven that quasi-varieties of normal bands are dualisable by finite alter egos and each finite normal band $\mathbf{D}$ is dualisable by a finite set of finitary relations. Hence by Theorem 2.4.1, $\mathbf{D}$ has (CLO), with respect to a finite set of relations. Therefore, we get the following corollary.

Corollary 3.2.11. *Every finite normal band has finite degree.*

3.3 An inherently non-dualisable algebra

In this section, we will show that the direct product of a dualisable algebra with a 2-element right-zero semigroup with constant is inherently non-dualisable. The 2-element implication algebra $\langle\{0,1\};\to,0\rangle$ with 0 as added constant is dualisable as it is term equivalent to the 2-element Boolean algebra [6, Exercise 10.6], while the 2-element right-zero semigroup with a constant is dualisable as it is a pointed set and so is covered by Banaschewski [3]. We recall the Inherently Non-Dualisable Algebra Theorem 2.4.6 to produce the following example.

Example 3.3.1. *Let* $\mathbf{I}_0 = \langle\{0,1\};\to,0\rangle$ *be the implication algebra with added constant and let* $\mathbf{R}_0 = \langle\{0,1\};\cdot,0\rangle$ *be a right-zero semigroup with* 0 *as added constant. The direct product* $\mathbf{I}_0 \times \mathbf{R}_0$ *is inherently non-dualisable.*

Proof. The direct product $\mathbf{I}_0 \times \mathbf{R}_0$ is isomorphic to the algebra

$$\mathbf{D} = \langle\{a,b,c,d\};*,a\rangle$$

with $*$ defined as follows.

$*$	a	b	c	d
a	b	b	d	d
b	a	b	c	d
c	b	b	d	d
d	a	b	c	d

We apply Theorem 2.4.6 to show that $\mathbf{D}$ is inherently non-dualisable. Let S be an infinite set and let $\mathbf{A}$ be the subalgebra of $\mathbf{D}^S$ with underlying set $D^S \backslash \{\underline{c}\}$. (We will leave it to the reader to check that $\mathbf{A}$ is indeed a subalgebra.) Let $A_0 = \{c_j^d \mid j \in S\}$ where c_j^d is defined to be the constant element $\underline{c}$ except with d in the jth coordinate. Let $u\colon \mathbb{N} \to \mathbb{N}$ be the function with $u(n) = 1$, for all n and let θ be a congruence on $\mathbf{A}$ with the index n. Assume that $c_i^d \; \theta \; c_j^d$ and $c_k^d \; \theta \; c_l^d$ with i, j, k, l pairwise unequal. Now we have $c_k^d = (c_i^d * c_i^d) * c_k^d \; \theta \; (c_j^d * c_i^d) * c_k^d = c_j^d{}_k^d$. By symmetry we get $c_j^d \; \theta \; c_j^d{}_k^d \; \theta \; c_k^d \; \theta \; c_l^d$. Hence $\theta\restriction_{A_0}$ has a unique block with more than $u(n)$ elements. It is easily checked that $\underline{c}$ is the ghost element. Since $\underline{c} \notin A$, we are done. □

Chapter 4

Natural duality based on the product of independent algebras

It is known that the direct product of two dualisable algebras need not be dualisable; see Examples 2.4.5 and 3.3.1. In this chapter, we examine the special case of independent varieties and show that in this case, the direct product of two dualisable algebras is indeed dualisable. The results in this chapter provide one half of the likely classification, of dualisability for completely simple semigroups; see Remark 4.2.8.

Independence of pair of algebras is a property first studied by Foster [20] and is reminiscent of the Chinese Remainder Theorem of elementary number theory. Later, Grätzer, Lakser, and Płonka [25] introduced the definition of independent *varieties*. They worked on joins and direct product of independent varieties and proved that independent subvarieties $\mathcal{V}_1$, $\mathcal{V}_2$ of $\mathcal{V}$ intersect at the trivial algebra and are such that their join $\mathcal{V}_1 \vee \mathcal{V}_2$ is $\mathbb{I}(\{\mathbf{A} \times \mathbf{B} \mid \mathbf{A} \in \mathcal{V}_1 \text{ and } \mathbf{B} \in \mathcal{V}_2\})$. Moreover, Knoebel [32] showed that the product of finitely based algebras is finitely based if these algebras are independent. Further results on independent varieties were obtained by Hu and Kelenson [29]. They proved that the congruence lattice of a member $\mathbf{A}$ from the smallest variety $\mathcal{K}$ containing independent varieties $\{\mathcal{K}_1, \dots, \mathcal{K}_n\}$

of the same type is factorized into a product of congruence lattices of some members $\mathbf{A}_i \in \mathcal{K}_i$. Also, Draškovičová [18] proved some results on the independence of varieties by considering congruence permutability of an algebra in the join of independent varieties.

4.1 Preliminaries

In this section we give a brief overview of independent varieties.

Definition 4.1.1. Let $\mathcal{K}$ and $\mathcal{L}$ be varieties of the same type. Then, $\mathcal{K}$ and $\mathcal{L}$ are *independent* if there is a binary term $*$ such that $\mathcal{K} \models x * y \approx x$ and $\mathcal{L} \models x * y \approx y$. We refer to $*$ as an *independence term* for $\mathcal{K}$ and $\mathcal{L}$.

Lemma 4.1.2. [20, Lemma 3.1.2] *A pair of varieties $\mathcal{K}$ and $\mathcal{L}$ are independent if and only if for all terms s and t, there exists a term u such that $\mathcal{K} \models s \approx u$ and $\mathcal{L} \models t \approx u$.*

Proof. ($\Rightarrow$) Assume that $\mathcal{K}$ and $\mathcal{L}$ are independent varieties. Then, by definition, there exists a binary term $*$ such that $\mathcal{K} \models x * y \approx x$ and $\mathcal{L} \models x * y \approx y$. Let s and t be terms and define $u = s * t$; then $\mathcal{K} \models u \approx s$ and $\mathcal{L} \models u \approx t$.
($\Leftarrow$) Define $s(x, y) := x$ and $t(x, y) := y$. Then by assumption there exists a term $u(x, y)$ such that $\mathcal{K} \models u \approx s$ and $\mathcal{L} \models u \approx t$, that is, $\mathcal{K} \models u(x, y) = x$ and $\mathcal{L} \models u(x, y) = y$. So, the term $x * y := u(x, y)$ witnesses the fact that $\mathcal{K}$ and $\mathcal{L}$ are independent. $\square$

We illustrate the previous definition and lemma by the following examples.

Example 4.1.3. *Let $\mathbf{G} = \langle G; \cdot \rangle$ be a finite group of order n and let $\mathbf{L}$ be a left-zero semigroup. Let $\mathcal{K} = \mathbb{HSP}(\mathbf{L})$ and $\mathcal{L} = \mathbb{HSP}(\mathbf{G})$. Then $x * y = x^n y$ shows that $\mathcal{K}$ and $\mathcal{L}$ are independent.*

Example 4.1.4. *Let* $\mathbf{2} = \langle \{0,1\}; \vee, \wedge \rangle$ *be a 2-element lattice and let* $\mathbf{R} = \langle R; \cdot, \cdot \rangle$ *be an algebra with* $\langle R; \cdot \rangle$ *a right-zero semigroup. Let* $\mathcal{K} = \mathbb{HSP}(\mathbf{2})$ *and* $\mathcal{L} = \mathbb{HSP}(\mathbf{R})$*. Then the term* $x * y = x \vee (x \wedge y)$ *shows that* $\mathcal{K}$ *and* $\mathcal{L}$ *are independent.*

Example 4.1.5. [25] *Let* $\mathcal{L}$ *be the class of all lattices* $\mathbf{M} = \langle M; \vee, \wedge \rangle$*. Let* $\mathcal{K}$ *be the class of all algebras* $\langle G; \vee, \wedge \rangle$ *such that* $\mathbf{G} = \langle G; \cdot, ^{-1} \rangle$ *is a group,* $x \vee y = xy$ *and* $x \wedge y = xy^{-1}$*. Then the term* $x * y = (x \vee y) \wedge y$ *establishes the independence of* $\mathcal{K}$ *and* $\mathcal{L}$*.*

4.2 Duality

In this section, we will prove that the direct product of two dualisable algebras that generate independent varieties is dualisable. We will build the duality of the product from the dualities of algebras that generate independent varieties. In order to prove the main result (Theorem 4.2.6), we need some preliminary results in the form of Lemmas 4.2.1–4.2.5. Throughout this chapter, we assume that $\mathbf{M}$ and $\mathbf{N}$ are finite algebras of the same type and that they generate independent varieties witnessed by an independence term $*$.

Lemma 4.2.1. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be algebras of the same type. Assume that* $\mathbf{M}$ *and* $\mathbf{N}$ *generate independent varieties witnessed by the independence term* $*$*. Then,*

(i) *every algebra in* $\mathbb{ISP}(\mathbf{M} \times \mathbf{N})$ *is isomorphic to* $\mathbf{A} \times \mathbf{B}$*, for some* $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ *and* $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$*.*

(ii) *the map* $*\colon (\mathbf{M} \times \mathbf{N})^2 \to \mathbf{M} \times \mathbf{N}$ *satisfies* $(a,b) * (c,d) = (a,d)$ *for all* $(a,b), (c,d) \in M \times N$*, that is,* $* = \pi_1^{\mathbf{M}\times\mathbf{N}} \times \pi_2^{\mathbf{M}\times\mathbf{N}}$*. Hence* $*$ *is a homomorphism.*

Proof. Part (i) is proved by Grätzer, Lakser, and Płonka [25, Theorem 1] and Taylor [46, Lemma 0.3], where (ii) is a trivial exercise. □

Remark 4.2.2. For any nonempty sets A, B, we endow $A \times B$ with the structure of a rectangular band, that is, for $(a,b),(c,d) \in A \times B$ we have

$$(a,b) \cdot (c,d) = (a,d).$$

Let $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ and $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$. Denote the set of homomorphisms from $\mathbf{A}$ to $\mathbf{M}$ by $D_{\mathbf{M}}(\mathbf{A})$, and define $D_{\mathbf{N}}(\mathbf{B})$ and $D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$ similarly. The operation $*$ is a binary homomorphism on $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{M}\times\mathbf{N}$. Hence $*$ is well defined on $D_{\mathbf{M}}(\mathbf{A})$, $D_{\mathbf{N}}(\mathbf{B})$ and $D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$. There is a natural map $\sigma\colon D_{\mathbf{M}}(\mathbf{A}) \times D_{\mathbf{N}}(\mathbf{B}) \to D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$ given by $\sigma(x,y) = x \times y$, that is $\sigma(x,y)(a,b) = (x(a), y(b))$, for all $(a,b) \in A \times B$.

Lemma 4.2.3. *Let $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ and let $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$. Then the map σ is an isomorphism with respect to the operation $*$. Indeed (with respect to $*$), $D_{\mathbf{M}}(\mathbf{A})$ is a left-zero semigroup, $D_{\mathbf{N}}(\mathbf{B})$ is a right-zero semigroup and $D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$ is a rectangular band isomorphic to $D_{\mathbf{M}}(\mathbf{A}) \times D_{\mathbf{N}}(\mathbf{B})$.*

Proof. It is clear that σ is a one-to-one map. We shall prove that σ is surjective. Let $z \in D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$. We want to find $x \in D_{\mathbf{M}}(\mathbf{A})$ and $y \in D_{\mathbf{N}}(\mathbf{B})$ such that $z = \sigma(x,y)$. Let $x\colon A \to M$ be the map given by $x_b = \pi_1(z(a,b))$ where $a \in A$ and b is a fixed element of B. We will show that x_b is independent of the choice of the element $b \in B$ from which it follows easily that x_b is homomorphism. Let $a \in A$ and $b, d \in B$, then

$\pi_1(z(a,b)), \pi_1(z(a,d)) \in M$. As $u * v = u$ in $\mathbf{M}$ and $u * v = v$ in $\mathbf{N}$, we have

$$\begin{aligned} x_b(a) = \pi_1(z(a,b)) &= \pi_1(z(a,b)) * \pi_1(z(a,d)) \\ &= \pi_1(z((a,b) * (a,d))) \qquad \text{as } \pi_1 \circ z \text{ is homomorphism} \\ &= \pi_1(z(a * a, b * d)) \\ &= \pi_1(z(a,d)) = x_d(a) \qquad \text{by independence.} \end{aligned}$$

Thus x_b is independent of choice. By symmetry, if we fix $a \in A$ and define $y_a \colon B \to N$ by $y_a(b) = \pi_2(z(a,b))$, then the map y_a is independent of the choice of a. Now choose any $c \in A$, $d \in B$ and define $x := x_d$ and $y := y_c$. Then for all $(a,b) \in A \times B$, we have

$$\begin{aligned} z(a,b) &= (\pi_1(z(a,b)), \pi_2(z(a,b))) \\ &= (x_b(a), y_a(b)) \\ &= (x \times y)(a,b) \\ &= \sigma(x,y)(a,b). \end{aligned}$$

Hence $z = \sigma(x,y)$, whence σ is surjective. Let $x_1, x_2 \in D_{\mathbf{M}}(\mathbf{A})$ and let $y_1, y_2 \in D_{\mathbf{N}}(\mathbf{B})$. we have

$$\begin{aligned} \sigma((x_1,y_1) * (x_2,y_2)) &= \sigma((x_1 * x_2, y_1 * y_2)) \\ &= \sigma((x_1, y_2)) \\ &= x_1 \times y_2 \\ &= (x_1 \times y_1) * (x_2 \times y_2) \qquad \text{by Lemma 4.2.1} \\ &= \sigma((x_1,y_1)) * \sigma((x_2,y_2)). \end{aligned}$$

Hence σ preserves $*$, so it is an isomorphism (with respects to $*$). Since $*$ is an independence term for $\mathbb{HSP}(\mathbf{M})$ and $\mathbb{HSP}(\mathbf{N})$, it is trivial that $\langle M, * \rangle$ is a left-zero semigroup and $\langle N, * \rangle$ is a right-zero semigroup. By Lemma

4.2.1, $\langle M\times N, *\rangle$ is a rectangular band. Since they are subsemigroups of the appropriate powers, it follows that $D_{\mathbf{M}}(\mathbf{A})$, $D_{\mathbf{N}}(\mathbf{B})$ and $D_{\mathbf{M}\times\mathbf{N}}(\mathbf{A}\times\mathbf{B})$ are respectively a left-zero semigroup, a right-zero semigroup and a rectangular band. □

The following lemma is effectively a special case of Lemma 4.2.3. (Define $\mathbf{M}$ to be the left-zero semigroup on M and $\mathbf{N}$ to be the right-zero semigroup on N.) Note that the homset $\mathcal{RB}(A\times B, M\times N)$ is a rectangular band.

Lemma 4.2.4. *Let A and B be subsets of M^k and N^k, respectively, for some $k\in\mathbb{N}$. Define $*$ on $A\times B$ and $M\times N$ to be the natural rectangular band operation. Then every map $\varphi\colon A\times B\to M\times N$ that preserves $*$ is of the form $\alpha\times\beta$ where $\alpha\colon A\to M$ and $\beta\colon B\to N$. Indeed,*

$$\sigma\colon M^A\times N^B\to\mathcal{RB}(A\times B, M\times N)$$

given by $\sigma(\alpha,\beta):=\alpha\times\beta$, is a bijection.

Let M, N and S be sets with M and N nonempty. Let $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$ be the natural projections and let $\rho_1\colon M^S\times N^S\to M^S$ and $\rho_2\colon M^S\times N^S\to N^S$ be the natural projections. There is a natural bijection

$$\mu\colon (M\times N)^S\to M^S\times N^S$$

given by $\mu(a)=(\pi_1\circ a,\pi_2\circ a)$, for all $a\in(M\times N)^S$. A subset C of $(M\times N)^S$ gives rise to subsets C_1 and C_2 of M^S and N^S, respectively, via $C_1=\rho_1(\mu(C))$ and $C_2=\rho_2(\mu(C))$. Given alter egos $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{N}}$ of $\mathbf{M}$ and $\mathbf{N}$, we now define a composite alter ego $\underset{\sim}{\mathbf{P}}$ of $\mathbf{P}=\mathbf{M}\times\mathbf{N}$. Let

$$\underset{\sim}{\mathbf{M}}=\langle M; G^{\underset{\sim}{\mathbf{M}}}, H^{\underset{\sim}{\mathbf{M}}}, R^{\underset{\sim}{\mathbf{M}}}, \mathfrak{T}\rangle \quad\text{and}\quad \underset{\sim}{\mathbf{N}}=\langle N; G^{\underset{\sim}{\mathbf{N}}}, H^{\underset{\sim}{\mathbf{N}}}, R^{\underset{\sim}{\mathbf{N}}}, \mathfrak{T}\rangle,$$

be alter egos of $\mathbf{M}$ and $\mathbf{N}$, respectively. For each n-ary total operation

$g^{\underset{\sim}{\mathbf{M}}} \in G^{\underset{\sim}{\mathbf{M}}}$, define $g^{\underset{\sim}{\mathbf{P}}} \in G^{\underset{\sim}{\mathbf{P}}}$ by

$$g^{\underset{\sim}{\mathbf{P}}}(((a_1, b_1), \dots, (a_n, b_n))) = (g^{\underset{\sim}{\mathbf{M}}}(a_1, \dots, a_n), b_1)$$

for all $((a_1, b_1), \dots, (a_n, b_n)) \in (M \times N)^n$. For each n-ary $g^{\underset{\sim}{\mathbf{N}}} \in G^{\underset{\sim}{\mathbf{N}}}$, the total operation $g^{\underset{\sim}{\mathbf{P}}} \in G^{\underset{\sim}{\mathbf{P}}}$ is defined symmetrically. For each n-ary partial operation $h^{\underset{\sim}{\mathbf{M}}} \in H^{\underset{\sim}{\mathbf{M}}}$, define

$$h^{\underset{\sim}{\mathbf{P}}} \colon \mu^{-1}(\operatorname{dom}(h^{\underset{\sim}{\mathbf{M}}}) \times N^n) \to M \times N \quad \text{by}$$

$$h^{\underset{\sim}{\mathbf{P}}}(((a_1, b_1), \dots, (a_n, b_n))) = (h^{\underset{\sim}{\mathbf{M}}}(a_1, \dots, a_n), b_1)$$

for all $(a_1, \dots, a_n) \in \operatorname{dom}(h^{\underset{\sim}{\mathbf{M}}})$ and $(b_1, \dots, b_n) \in N^n$. Again, for each n-ary $h^{\underset{\sim}{\mathbf{N}}} \in H^{\underset{\sim}{\mathbf{N}}}$, the partial operation $h^{\underset{\sim}{\mathbf{P}}}$ is defined symmetrically. The relation $r^{\underset{\sim}{\mathbf{P}}} \in R^{\underset{\sim}{\mathbf{P}}}$ is defined by $r^{\underset{\sim}{\mathbf{P}}} := \mu^{-1}(r^{\underset{\sim}{\mathbf{M}}} \times N^n)$ where $r^{\underset{\sim}{\mathbf{M}}} \in R^{\underset{\sim}{\mathbf{M}}}$. Symmetrically, we define $r^{\underset{\sim}{\mathbf{P}}}$ for each n-ary relation $r^{\underset{\sim}{\mathbf{N}}} \in R^{\underset{\sim}{\mathbf{N}}}$. We can now define the alter ego

$$\underset{\sim}{\mathbf{P}} = \langle P; *, G^{\underset{\sim}{\mathbf{P}}}, H^{\underset{\sim}{\mathbf{P}}}, R^{\underset{\sim}{\mathbf{P}}}, \mathcal{T} \rangle$$

where

$$G^{\underset{\sim}{\mathbf{P}}} = \{ g^{\underset{\sim}{\mathbf{P}}} \mid g \in G^{\underset{\sim}{\mathbf{M}}} \cup G^{\underset{\sim}{\mathbf{N}}} \},$$

$$H^{\underset{\sim}{\mathbf{P}}} = \{ h^{\underset{\sim}{\mathbf{P}}} \mid h \in H^{\underset{\sim}{\mathbf{M}}} \cup H^{\underset{\sim}{\mathbf{N}}} \},$$

$$R^{\underset{\sim}{\mathbf{P}}} = \{ r^{\underset{\sim}{\mathbf{P}}} \mid r \in R^{\underset{\sim}{\mathbf{M}}} \cup R^{\underset{\sim}{\mathbf{N}}} \}$$

and $\mathcal{T}$ is the discrete topology.

Lemma 4.2.5. *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G^{\underset{\sim}{\mathbf{M}}}, H^{\underset{\sim}{\mathbf{M}}}, R^{\underset{\sim}{\mathbf{M}}}, \mathcal{T} \rangle$ *and* $\underset{\sim}{\mathbf{N}} = \langle N; G^{\underset{\sim}{\mathbf{N}}}, H^{\underset{\sim}{\mathbf{N}}}, R^{\underset{\sim}{\mathbf{N}}}, \mathcal{T} \rangle$ *be alter egos of* $\mathbf{M}$ *and* $\mathbf{N}$*, respectively. Define* $\mathbf{P} = \mathbf{M} \times \mathbf{N}$ *and let* $\underset{\sim}{\mathbf{P}}$ *be the alter ego of* $\mathbf{P}$ *as defined above. Then, for all* $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ *and* $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$*, and for every* $\underset{\sim}{\mathbf{P}}$*-morphism* $\gamma \colon D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B}) \to \underset{\sim}{\mathbf{P}}$ *there exist* $\underset{\sim}{\mathbf{M}}$*- and* $\underset{\sim}{\mathbf{N}}$*-morphisms,* $\alpha \colon D_{\mathbf{M}}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *and* $\beta \colon D_{\mathbf{N}}(\mathbf{B}) \to \underset{\sim}{\mathbf{N}}$*, respectively, such that*

$\gamma \circ \sigma = \alpha \times \beta$ *where* σ *is the map defined immediately before Lemma 4.2.3.*

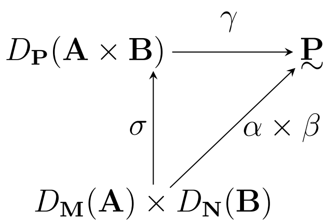

Figure 4.1: Commutative diagram for σ, γ and $\alpha \times \beta$

Proof. Let $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ and $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$. Let $\gamma \colon D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B}) \to \underset{\sim}{\mathbf{P}}$ be a $\underset{\sim}{\mathbf{P}}$-morphism. As $*$ is in the type of $\underset{\sim}{\mathbf{P}}$ and γ is a $\underset{\sim}{\mathbf{P}}$-morphism, γ preserves the independence term $*$. Therefore, by Lemma 4.2.3 we get $\gamma \circ \sigma = \alpha \times \beta$, for some *maps* $\alpha \colon D_{\mathbf{M}}(\mathbf{A}) \to M$ and $\beta \colon D_{\mathbf{N}}(\mathbf{B}) \to N$. We shall prove that α and β are $\underset{\sim}{\mathbf{M}}$- and $\underset{\sim}{\mathbf{N}}$-morphisms, respectively. Let $r \in R^{\underset{\sim}{\mathbf{M}}}$ be an n-ary relation and recall that $r \in R^{\underset{\sim}{\mathbf{M}}}$ gives rise to a relation in $r \in R^{\underset{\sim}{\mathbf{P}}}$, which we shall also denote simply by r. Let $x_1, \dots, x_n \in D_{\mathbf{M}}(\mathbf{A})$, $y \in D_{\mathbf{N}}(\mathbf{B})$. Then

$$
\begin{aligned}
& (x_1, \dots, x_n) \in r \quad \text{in } D_{\mathbf{M}}(\mathbf{A}) \\
\Longrightarrow\ & (x_1 \times y, \dots, x_n \times y) \in r \quad \text{in } D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B}) \\
\Longrightarrow\ & (\gamma(x_1 \times y), \dots, \gamma(x_n \times y)) \in r \\
\Longrightarrow\ & ((\gamma \circ \sigma)(x_1, y), \dots, (\gamma \circ \sigma)(x_n, y)) \in r \\
\Longrightarrow\ & ((\alpha \times \beta)(x_1, y), \dots, (\alpha \times \beta)(x_n, y)) \in r \\
\Longrightarrow\ & ((\alpha(x_1), \beta(y)), \dots, (\alpha(x_n), \beta(y))) \in r \\
\Longrightarrow\ & (\alpha(x_1), \dots, \alpha(x_n)) \in r \quad \text{in } \underset{\sim}{\mathbf{M}}.
\end{aligned}
$$

Similarly, assume that $(x_1, \dots, x_n) \in \operatorname{dom}(h^{\underset{\sim}{\mathbf{M}}})$ in $D_{\mathbf{M}}(\mathbf{A})$ and recall that the partial operation $h \in H^{\underset{\sim}{\mathbf{M}}}$ gives rise to partial operation $h \in H^{\underset{\sim}{\mathbf{P}}}$ which we shall denote simply by h. Then

$$\begin{aligned} & (x_1 \times y, \ldots, x_n \times y) \in \mathrm{dom}(h) \quad \text{in } D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B}) \\ \Longrightarrow\ & (\gamma(x_1 \times y), \ldots, \gamma(x_n \times y)) \in \mathrm{dom}(h) \\ \Longrightarrow\ & ((\gamma \circ \sigma)(x_1, y), \ldots, (\gamma \circ \sigma)(x_n, y)) \in \mathrm{dom}(h) \\ \Longrightarrow\ & ((\alpha \times \beta)(x_1, y), \ldots, (\alpha \times \beta)(x_n, y)) \in \mathrm{dom}(h) \\ \Longrightarrow\ & ((\alpha(x_1), \beta(y)), \ldots, (\alpha(x_n), \beta(y))) \in \mathrm{dom}(h) \\ \Longrightarrow\ & (\alpha(x_1), \ldots, \alpha(x_n)) \in \mathrm{dom}(h) \quad \text{in } \utilde{\mathbf{M}}. \end{aligned}$$

Moreover, for $h \in H^{\utilde{\mathbf{M}}}$ we have $\alpha(h(x_1, \ldots, x_n)) = h(\alpha(x_1), \ldots, \alpha(x_n))$ since

$$\begin{aligned} (h(\alpha(x_1), \ldots, \alpha(x_n)), \beta(y)) &= h((\alpha(x_1), \beta(y)), \ldots, (\alpha(x_n), \beta(y))) \\ &= h((\alpha \times \beta)(x_1, y), \ldots, (\alpha \times \beta)(x_n, y)) \\ &= h((\gamma \circ \sigma)(x_1, y), \ldots, (\gamma \circ \sigma)(x_n, y)) \\ &= h(\gamma(x_1 \times y), \ldots, \gamma(x_n \times y)) \\ &= \gamma(h(x_1 \times y, \ldots, x_n \times y)) \\ &= \gamma(h(x_1, \ldots, x_n) \times y) \\ &= (\gamma \circ \sigma)(h(x_1, \ldots, x_n), y) \\ &= (\alpha \times \beta)(h(x_1, \ldots, x_n), y) \\ &= (\alpha(h(x_1, \ldots, x_n)), \beta(y)). \end{aligned}$$

Note that the case of the total operations is a particular case of the partial operation argument. Hence α is an $\utilde{\mathbf{M}}$-morphism. By symmetry, β is an $\utilde{\mathbf{N}}$-morphism. □

Theorem 4.2.6. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite algebras that generate independent varieties witnessed by the independence term* $*$. *Assume that* $\mathbf{M}$ *and* $\mathbf{N}$ *are* *dualised by the alter egos*

$$\underset{\sim}{\mathbf{M}} = \langle M; R^{\underset{\sim}{\mathbf{M}}}, \mathcal{T}\rangle \quad \textit{and} \quad \underset{\sim}{\mathbf{N}} = \langle N; R^{\underset{\sim}{\mathbf{N}}}, \mathcal{T}\rangle,$$

respectively. Then $\mathbf{P} := \mathbf{M} \times \mathbf{N}$ *is dualised by the alter ago*

$$\underset{\sim}{\mathbf{P}} = \langle P; *, R^{\underset{\sim}{\mathbf{P}}}, \mathcal{T}\rangle,$$

where, for $k \in \mathbb{N}$*,*

$$R^{\underset{\sim}{\mathbf{P}}} = \{\mu^{-1}(r \times N^k) \mid r \in R^{\underset{\sim}{\mathbf{M}}}\} \cup \{\mu^{-1}(M^k \times r) \mid r \in R^{\underset{\sim}{\mathbf{N}}}\}.$$

Proof. Let $\mathbf{C} \in \mathbb{ISP}(\mathbf{P})$. By Lemma 4.2.1, we may assume that $\mathbf{C} \cong \mathbf{A} \times \mathbf{B}$ for some $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ and $\mathbf{B} \in \mathbb{ISP}(\mathbf{N})$. By Lemma 4.2.3, for all $z \in D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B})$, there exist $x \in D_{\mathbf{M}}(\mathbf{A})$ and $y \in D_{\mathbf{N}}(\mathbf{B})$ such that $z = \sigma(x, y)$. Let $\gamma\colon D_{\mathbf{P}}(\mathbf{A} \times \mathbf{B}) \to \underset{\sim}{\mathbf{P}}$ be a $\underset{\sim}{\mathbf{P}}$-morphism. We want to prove that there exists $(a, b) \in A \times B$ such that $\gamma(z) = z((a, b))$. As $*$ is in the type of $\underset{\sim}{\mathbf{P}}$ and γ is a $\underset{\sim}{\mathbf{P}}$-morphism, then γ preserves the independence term $*$. Therefore, by Lemma 4.2.5 we get $\gamma \circ \sigma = \alpha \times \beta$ for some morphisms $\alpha\colon D_{\mathbf{M}}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ and $\beta\colon D_{\mathbf{N}}(\mathbf{B}) \to \underset{\sim}{\mathbf{N}}$. Since $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{N}}$ dualise $\mathbf{M}$ and $\mathbf{N}$, respectively, there exists $a \in A$ such that $\alpha(x) = x(a)$, for all $x \in D_{\mathbf{M}}(\mathbf{A})$, and there exists $b \in B$ such that $\beta(y) = y(b)$, for all $y \in D_{\mathbf{N}}(\mathbf{B})$. Thus

$$\begin{aligned}\gamma(z) = (\gamma \circ \sigma)(x, y) = (\alpha(x), \beta(y)) = (x(a), y(b))& \\ = (x \times y)(a, b) = \sigma(x, y)(a, b) = z((a, b)).&\end{aligned}$$

Therefore, $\underset{\sim}{\mathbf{P}}$ dualises $\mathbf{P} = \mathbf{M} \times \mathbf{N}$. □

By combining Examples 4.1.3–4.1.5 and Theorem 4.2.6, we obtain several examples of dualisable semigroups.

Corollary 4.2.7. *Let* **G** *be a finite dualisable group,* **L** *be a finite left-zero semigroup,* **R** *be a finite right-zero semigroup and* **2** *be a 2-element lattice. Then*

(1) $\mathbf{L} \times \mathbf{R}$ *is dualisable;*

(2) $\mathbf{G} \times \mathbf{L}$ *is dualisable;*

(3) $\mathbf{G} \times \mathbf{R}$ *is dualisable;*

(4) $\mathbf{L} \times \mathbf{2}$ *is dualisable;*

(5) $\mathbf{G} \times \mathbf{L} \times \mathbf{R}$ *is dualisable.*

Proof. By Examples 4.1.3–4.1.5 and Theorem 4.2.6, $(1)-(4)$ are dualisable. Since $\mathbf{G} \times \mathbf{L}$ is dualisable and **G**, **L** and **R** generate independent varieties, then there exists a term $x * y = x \cdot y^n$ showed that the varieties generated by $\mathbf{G} \times \mathbf{L}$ and **R** are independent where x is the term satisfied on $\mathbf{G} \times \mathbf{L}$ and n is the order of the group **G**. Hence $\mathbf{G} \times \mathbf{L} \times \mathbf{R}$ is dualisable. □

Remark 4.2.8. It is known that the following are equivalent for a finite completely simple semigroup **M**.

(1) The variety $\mathbb{HSP}(\mathbf{M})$ is residually finite ([23], [35], [42]).

(2) The quasi-variety $\mathbb{ISP}(\mathbf{M})$ is finitely based [41].

(3) $\mathbf{M} \cong \mathbf{RB} \times \mathbf{G}$ where **RB** is rectangular band and **G** is a group whose Sylow subgroups are abelian.

Quackenbush and Szabó [39] have conjectured that a finite group with abelian Sylow subgroups is dualisable and they proved this in the case when all Sylow subgroups are cyclic [40]. A preprint claiming to prove the rest of the conjecture has been circulated, but the paper has not yet been accepted. If the *Quackenbush-Szabó conjecture* is true, then Corollary 4.2.7 shows that the semigroups in item (3) of Remark 4.2.8 above are dualisable. We conjecture the converse 'if a finite completely simple semigroup is dualisable, it has a form as in (3)' also holds.

Chapter 5

Generator independence for (IC)

Let $\mathbf{M}$ and $\mathbf{D}$ be finite algebras that generate the same quasi-variety. The Independence of the Generator Theorem states that if $\mathbf{D}$ is dualisable then $\mathbf{M}$ is also dualisable. It is one of the fundamental results in natural duality theory and was proved independently by Saramago [43] and Davey and Willard [17]. Hyndman [30] extended this result by showing that if $\mathbf{D}$ is strongly dualisable then $\mathbf{M}$ is also. In [9], Davey and Haviar find an efficient and user friendly transfer of a strong duality for the quasi-variety $\mathcal{A}$ based on $\mathbf{D}$ to a strong duality based on $\mathbf{M}$. More recently, these results have been extended to multisorted dualities for quasi-varieties generated by a finite set of finite algebras: in this setting Davey, Gouveia, Haviar and Priestley [8] proved dualisability is independent of the generating set and Davey, Haviar and Pitkethly [10] proved the corresponding results for full and strong dualisability.

In this chapter, we prove that if $\mathbf{D}$ has an alter ego of finite type that satisfies the Interpolation Condition (IC), then $\mathbf{M}$ does as well. We can always find an alter ego $\underset{\sim}{\mathbf{M}}$ of $\mathbf{M}$ that satisfies (IC); simply choose $\underset{\sim}{\mathbf{M}}$ to be the brute-force alter ego (see The Brute Force Duality Theorem 2.3.1, [6]). The issue here is to take an alter ego $\underset{\sim}{\mathbf{D}}$ of finite type and transfer it explicitly to an alter ego $\underset{\sim}{\mathbf{M}}$ of finite type. In that case, the former alter ego $\underset{\sim}{\mathbf{M}}$ will yield a duality on the quasi-variety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$. Corresponding results for dualisability, rather than (IC), were proved by Gouveia and Haviar [24].

5.1 (IC) Transfer

The following theorem is the main theorem of this chapter. The proof will be via Lemmas 5.1.2 and 5.1.3.

Theorem 5.1.1. *Let* $\mathbf{M}$ *and* $\mathbf{D}$ *be finite algebras that generate the same quasi-variety. If* $\mathbf{D}$ *has an alter ego of finite type that satisfies (IC), then so does* $\mathbf{M}$.

Throughout, we let $\mathbf{M}$ and $\mathbf{D}$ be finite algebras that generate the same quasi-variety. Let $\{\alpha_1, \dots, \alpha_k\}$ be a set of separating homomorphisms from $\mathbf{M}$ to $\mathbf{D}$. Let

$$\alpha := \alpha_1 \sqcap \cdots \sqcap \alpha_k \colon M \to D^k$$

be the natural product map and let $M_\alpha = \alpha(M)$. Then $\alpha_i = \pi_i \circ \alpha$ where $i \leq k$ and π_i is the ith projection. Hence M_α is a k-ary algebraic relation on $\mathbf{D}$. Assume that $\beta \colon \mathbf{D} \hookrightarrow \mathbf{M}$ is an embedding and let $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$ be an alter ego of $\mathbf{D}$. We will interpret the operations in G, the partial operations in H and the relations in R on $\mathbf{M}$. For every n-ary operation $g \colon D^n \to D$ in G, we define $g_\beta \colon (\beta(D))^n \to M$ by

$$g_\beta(\beta(x_1), \dots, \beta(x_n)) = \beta(g(x_1, \dots, x_n)),$$

for all $(x_1, \dots, x_n) \in D^n$ (well defined as β is an embedding). Observe that g_β is a homomorphism from the subalgebra $(\beta(D))^n$ of $\mathbf{M}^n$ to $\mathbf{M}$. For every n-ary partial operation $h \in H$

$$h \colon \operatorname{dom}(h) \to D$$

in H, let

$$\mathrm{dom}(h_\beta) = \{(\beta(x_1), \dots, \beta(x_n)) \mid (x_1, \dots, x_n) \in \mathrm{dom}(h)\} = \beta^n(\mathrm{dom}(h)) \subseteq M^n.$$

Let $h_\beta \colon \mathrm{dom}(h_\beta) \to M$ be the map defined by

$$h_\beta(\beta(x_1), \dots, \beta(x_n)) = \beta(h(x_1, \dots, x_n)).$$

Then h_β is well defined as β is an embedding, and a homomorphism from the subalgebra $\mathrm{dom}(h_\beta)$ of $\mathbf{M}^n$ into $\mathbf{M}$. Let $r \in R$ and let

$$r_\beta = \{(\beta(x_1), \dots, \beta(x_n)) \mid (x_1, \dots, x_n) \in r\}.$$

Hence r_β is a subalgebra of $\mathbf{M}^n$. For each $i \in \{1, \dots, k\}$, the homomorphism $\gamma_i = \beta \circ \alpha_i$ is an endomorphism of $\mathbf{M}$. Define

$$\Gamma_{\alpha\beta} := \{\gamma_i \mid i \in \{1, \dots, k\}\}.$$

Lemma 5.1.2. *Assume that* $\mathbf{M}$ *and* $\mathbf{D}$ *generate the same quasi-variety with* $\mathbf{D} \in \mathbb{IS}(\mathbf{M})$*, and let* α *and* β *be as specified above. Let* $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ *be an alter ego of* $\mathbf{D}$ *that satisfies (IC). Then* $\underset{\sim}{\mathbf{M}} = \langle M; \Gamma_{\alpha\beta}, G_\beta \cup H_\beta, R_\beta, \mathcal{T}\rangle$ *satisfies (IC), where*

$$G_\beta = \{g_\beta \mid g \in G\}, \quad H_\beta = \{h_\beta \mid h \in H\} \quad \text{and} \quad R_\beta = \{r_\beta \mid r \in R\} \cup \{\beta(D)\}.$$

Proof. Let $\mathbf{X} \leqslant \underset{\sim}{\mathbf{M}}^m$ for some natural number m. Consider a morphism $\varphi \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$. We want to prove that φ extends to some term function of $\mathbf{M}$. Let $Y = (\beta^m)^{-1}(X)$. Since $\Gamma_{\alpha\beta} \neq \varnothing$, it follows that $Y \neq \varnothing$ provided that $X \neq \varnothing$. In addition, since $\mathbf{X}$ is a substructure of $\underset{\sim}{\mathbf{M}}^m$, it follows easily that $\mathbf{Y}$ is a substructure of $\underset{\sim}{\mathbf{D}}^m$. Let $\widehat{\varphi}\colon Y \to D$ be the map defined by

$$\widehat{\varphi}(y) = \beta^{-1}(\varphi(\beta^m(y))).$$

Note that $\widehat{\varphi}$ is well defined as φ preserves $\beta(D)$. We claim that $\widehat{\varphi}$ is a $\underset{\sim}{\mathbf{D}}$-morphism. Let $y_1, \ldots, y_n \in Y$ and $(y_1, \ldots, y_n) \in r$ where $r \in R$. Then,

$$\begin{aligned} & (\beta^m(y_1), \ldots, \beta^m(y_n)) \in r_\beta^{\mathbf{X}} \\ \Rightarrow\quad & (\varphi(\beta^m(y_1)), \ldots, \varphi(\beta^m(y_n))) \in r_\beta \\ \Rightarrow\quad & (\beta^{-1}(\varphi(\beta^m(y_1))), \ldots, \beta^{-1}(\varphi(\beta^m(y_n)))) \in r \\ \Rightarrow\quad & (\widehat{\varphi}(y_1), \ldots, \widehat{\varphi}(y_n)) \in r. \end{aligned}$$

Let $h \in G \cup H$ and let $y_1, \ldots, y_n \in Y$ and $(y_1, \ldots, y_n) \in \mathrm{dom}(h^{\mathbf{Y}})$. Then,

$$\begin{aligned} & (\beta^m(y_1), \ldots, \beta^m(y_n)) \in \mathrm{dom}(h_\beta^{\mathbf{X}}) \\ \Rightarrow\quad & (\varphi(\beta^m(y_1)), \ldots, \varphi(\beta^m(y_n))) \in \mathrm{dom}(h_\beta) \\ \Rightarrow\quad & (\beta^{-1}(\varphi(\beta^m(y_1))), \ldots, \beta^{-1}(\varphi(\beta^m(y_n)))) \in \mathrm{dom}(h) \\ \Rightarrow\quad & (\widehat{\varphi}(y_1), \ldots, \widehat{\varphi}(y_n)) \in \mathrm{dom}(h). \end{aligned}$$

Also,

$$\begin{aligned} h((\widehat{\varphi}(y_1), \ldots, \widehat{\varphi}(y_n))) &= h(\beta^{-1}(\varphi(\beta^m(y_1))), \ldots, \beta^{-1}(\varphi(\beta^m(y_n)))) \\ &= \beta^{-1}(h_\beta(\varphi(\beta^m(y_1)), \ldots, \varphi(\beta^m(y_n)))) \\ &= \beta^{-1}(\varphi(h_\beta(\beta^m(y_1), \ldots, \beta^m(y_n)))) \\ &= \beta^{-1}(\varphi(\beta^m(h(y_1, \ldots, y_n)))) \\ &= \widehat{\varphi}(h(y_1, \ldots, y_n)). \end{aligned}$$

Therefore, $\widehat{\varphi}$ is a morphism from $\mathbf{Y}$ to $\underset{\sim}{\mathbf{D}}$. Since $\underset{\sim}{\mathbf{D}}$ satisfies (IC), there exists an m-ary term t such that $\widehat{\varphi}(y) = t^{\mathbf{D}}(y)$, for all $y \in Y$. Let $x \in X$; then for all $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned}\gamma_i^{\mathbf{X}}(x_1, \ldots, x_m) &= (\gamma_i(x_1), \ldots, \gamma_i(x_m)) \\ &= (\beta(\alpha_i(x_1)), \ldots, \beta(\alpha_i(x_m))) \qquad (\dagger) \\ &= (\beta^m(\alpha_i(x_1), \ldots, \alpha_i(x_m))).\end{aligned}$$

Then, we have

$$\begin{aligned}\alpha_i(\varphi(x)) &= \beta^{-1}(\beta(\alpha_i(\varphi(x)))) \\ &= \beta^{-1}(\gamma_i(\varphi(x_1, \ldots, x_m))) \\ &= \beta^{-1}(\varphi(\gamma_i^{\mathbf{X}}(x_1, \ldots, x_m))) \\ &= \beta^{-1}(\varphi(\beta^m(\alpha_i(x_1), \ldots, \alpha_i(x_m)))) \quad \text{by } \dagger \\ &= \widehat{\varphi}(\alpha_i(x_1), \ldots, \alpha_i(x_m)) \\ &= t^{\mathbf{D}}(\alpha_i(x_1), \ldots, \alpha_i(x_m)) \\ &= \alpha_i(t^{\mathbf{M}}(x_1, \ldots, x_m)).\end{aligned}$$

Hence $\alpha_i(\varphi(x)) = \alpha_i(t^{\mathbf{M}}(x))$. Since $\{\alpha_i \mid i \in \{1, \ldots, k\}\}$ separates the points of $\mathbf{M}$, it follows that $\varphi(x) = t^{\mathbf{M}}(x)$, whence φ extends to the term function $t^{\mathbf{M}}$. Therefore, the alter ego $\underset{\sim}{\mathbf{M}}$ satisfies (IC). □

Let $\mathbf{M}$ be a finite algebra and let $\mathbf{N} = \mathbf{M}^2$. Assume that

$$\underset{\sim}{\mathbf{N}} = \langle N; G, H, R, \mathfrak{T}\rangle$$

is an alter ego of $\mathbf{N}$ that satisfies (IC). For every n-ary operation $g \in G$ and for $i \in \{1, 2\}$, let $\pi_i \colon N \to M$ be the natural projections and let $g_i^M \colon M^{2n} \to M$ be defined by

$$g_i^M(a_1, \ldots, a_n, b_1, \ldots, b_n) = \pi_i\Big(g((a_1, b_1), \ldots, (a_n, b_n))\Big) \text{ for } i \in \{1, 2\}.$$

For an n-ary partial operation $h \in H$ and $i \in \{1,2\}$, let

$$\mathrm{dom}(h_i^M) = \{(a_1, \dots, a_n, b_1, \dots, b_n) \in M^{2n} \mid ((a_1, b_1), \dots, (a_n, b_n)) \in \mathrm{dom}(h)\}.$$

Let $h_i^M \colon \mathrm{dom}(h_i^M) \to M$ be defined by

$$h_i^M(a_1, \dots, a_n, b_1, \dots, b_n) = \pi_i\Big(h((a_1, b_1), \dots, (a_n, b_n))\Big).$$

Let $r \in R$ be an n-ary relation and define the $2n$-ary relation on M by

$$r^M = \{(a_1, \dots, a_n, b_1, \dots, b_n) \in M^{2n} \mid \big((a_1, b_1), \dots, (a_n, b_n)\big) \in r\}.$$

Let

$$G_M = \{g_i^M \mid i \in \{1,2\} \text{ and } g \in G\} \cup \{\pi_i\},$$

$$H_M = \{h_i^M \mid i \in \{1,2\} \text{ and } h \in H\} \text{ and}$$

$$R_M = \{r^M \mid r \in R\}.$$

Let $\underset{\sim}{\mathbf{M}} = \langle M; G_M, H_M, R_M, \mathfrak{T}\rangle$.

Lemma 5.1.3. *Let* $\mathbf{M}$ *be a finite algebra and let* $\mathbf{N} = \mathbf{M}^2$. *Assume that* $\underset{\sim}{\mathbf{N}} = \langle N; G, H, R, \mathfrak{T}\rangle$ *is an alter ego of* $\mathbf{N}$ *that satisfies (IC). Then* $\underset{\sim}{\mathbf{M}} = \langle M; G_M, H_M, R_M, \mathfrak{T}\rangle$ *satisfies (IC).*

Proof. Let $\mathbf{X} \leqslant \underset{\sim}{\mathbf{M}}^m$ for some $m \in \mathbb{N}$. Let $\varphi \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ be a morphism. Let

$$Y = \{((x_1, y_1), \dots, (x_m, y_m)) \mid (x_1, \dots, x_m), (y_1, \dots, y_m) \in X\} \subseteq N^m$$

We claim that Y is a substructure of $\underset{\sim}{\mathbf{N}}^m$. Let $\rho_i \colon N^m \to M^m$ for $i \in \{1,2\}$ be the natural projections. Let $y_1, \dots, y_n \in Y$, then

$$\big(\rho_1(y_1), \dots, \rho_1(y_n), \rho_2(y_1), \dots, \rho_2(y_n)\big) \in \mathrm{dom}(h_i^M)$$

then $(y_1, \ldots, y_n) \in \mathrm{dom}(h)$ where $h \in G \cup H$ and

$$h(y_1, \ldots, y_n) = \Big(h_1^M\big(\rho_1(y_1), \ldots, \rho_1(y_n), \rho_2(y_1), \ldots, \rho_2(y_n)\big), \\ h_2^M\big(\rho_1(y_1), \ldots, \rho_1(y_n), \rho_2(y_1), \ldots, \rho_2(y_n)\big)\Big).$$

Since

$$h_i^M\big(\rho_1(y_1), \ldots, \rho_1(y_n), \rho_2(y_1), \ldots, \rho_2(y_n)\big) \in X,$$

it follows that $h(y_1, \ldots, y_n) \in Y$. Therefore, $\mathbf{Y}$ is a substructure of $\underset{\sim}{\mathbf{N}}^m$. Let $\widehat{\varphi}\colon Y \to N$ be the map defined by

$$\widehat{\varphi}(y) = \big(\varphi(\rho_1(y)), \varphi(\rho_2(y))\big).$$

We claim that $\widehat{\varphi}$ preserves G, H and R. Let $h \in G \cup H$, let $y_1, \ldots, y_n \in Y$ and let $(y_1, \ldots, y_n) \in \mathrm{dom}(h)$. Then

$$\begin{aligned} & \big(\rho_1(y_1), \ldots, \rho_1(y_n), \rho_2(y_1), \ldots, \rho_2(y_n)\big) \in \mathrm{dom}(h_i^M) \\ \Rightarrow\quad & \big(\varphi(\rho_1(y_1)), \ldots, \varphi(\rho_1(y_n)), \varphi(\rho_2(y_1)), \ldots, \varphi(\rho_2(y_n))\big) \in \mathrm{dom}(h_i^M) \\ \Rightarrow\quad & \Big(\big(\varphi(\rho_1(y_1)), \varphi(\rho_2(y_1))\big), \ldots, \big(\varphi(\rho_1(y_n)), \varphi(\rho_2(y_n))\big)\Big) \in \mathrm{dom}(h) \\ \Rightarrow\quad & \big(\widehat{\varphi}(y_1), \ldots, \widehat{\varphi}(y_n)\big) \in \mathrm{dom}(h). \end{aligned}$$

We have

$$\begin{aligned}
h\Big(\widehat{\varphi}(y_1),\ldots,\widehat{\varphi}(y_n)\Big) &= h\Big(\big(\varphi(\rho_1(y_1)),\varphi(\rho_2(y_1))\big),\ldots,\big(\varphi(\rho_1(y_n)),\varphi(\rho_2(y_n))\big)\Big)\\
&= \Big(h_1^M\big(\varphi(\rho_1(y_1)),\ldots,\varphi(\rho_1(y_n)),\varphi(\rho_2(y_1)),\ldots,\varphi(\rho_2(y_n))\big),\\
&\qquad h_2^M\big(\varphi(\rho_1(y_1)),\ldots,\varphi(\rho_1(y_n)),\varphi(\rho_2(y_1)),\ldots,\varphi(\rho_2(y_n))\big)\Big)\\
&= \Big(\varphi\big(h_1^M(\rho_1(y_1),\ldots,\rho_1(y_n),\rho_2(y_1),\ldots,\rho_2(y_n))\big),\\
&\qquad \varphi\big(h_2^M(\rho_1(y_1),\ldots,\rho_1(y_n),\rho_2(y_1),\ldots,\rho_2(y_n))\big)\Big)\\
&= \Big(\varphi\big(\rho_1(h(y_1,\ldots,y_n))\big),\varphi\big(\rho_2(h(y_1,\ldots,y_n))\big)\Big)\\
&= \widehat{\varphi}\Big(h(y_1,\ldots,y_n)\Big).
\end{aligned}$$

Let $(y_1,\ldots,y_n)\in r$. Then

$$\begin{aligned}
&\Big(\rho_1(y_1),\ldots,\rho_1(y_n),\rho_2(y_1),\ldots,\rho_2(y_n)\Big)\in r^M\\
\Rightarrow\quad &\Big(\varphi(\rho_1(y_1)),\ldots,\varphi(\rho_1(y_n)),\varphi(\rho_2(y_1)),\ldots,\varphi(\rho_2(y_n))\Big)\in r^M\\
\Rightarrow\quad &\Big(\big(\varphi(\rho_1(y_1)),\varphi(\rho_2(y_1))\big),\ldots,\big(\varphi(\rho_1(y_n)),\varphi(\rho_2(y_n))\big)\Big)\in r\\
\Rightarrow\quad &\Big(\widehat{\varphi}(y_1),\ldots,\widehat{\varphi}(y_n)\Big)\in r.
\end{aligned}$$

Hence $\widehat{\varphi}$ is a morphism from the substructure $\mathbf{Y}$ of $\underset{\sim}{\mathbf{N}}^m$ into $\underset{\sim}{\mathbf{N}}$. Therefore, for some term t , we have $\widehat{\varphi}(y)=t^{\mathbf{N}}(y)$ for all $y\in Y$. Let $x=(x_1,\ldots,x_m)\in X$ and let $y=((x_1,x_1),\ldots,(x_m,x_m))\in Y$. Then

$$\begin{aligned}
\varphi(x) &= \pi_1(\widehat{\varphi}(y))\\
&= \pi_1(t^{\mathbf{N}}(y))\\
&= t^{\mathbf{M}}(\pi_1(y))\\
&= t^{\mathbf{M}}(x).
\end{aligned}$$

Hence φ extends to the term function $t^{\mathbf{M}}$ on $\mathbf{M}$. □

We now prove the main Theorem 5.1.1. Let **M** and **D** be finite type algebras that generate the same quasi-variety and assume that the alter ego $\underset{\sim}{\mathbf{D}}$ satisfies (IC) with $\underset{\sim}{\mathbf{D}}$ of finite type. So, for some n, we have **D** embeds in $\mathbf{M}^{2^n}$ and then Lemma 5.1.2, provides an alter ego of $\mathbf{M}^{2^n}$ of finite type satisfying (IC). Finally, repeated application of Lemma 5.1.3, yields an alter ego of **M** of finite type that satisfies (IC).

Chapter 6

Dualisability of some Clifford semigroups

A Clifford semigroup is a completely regular semigroup with commuting idempotents. As an inverse semigroup is a regular semigroup with its idempotents commuting, so a Clifford semigroup is a completely regular inverse semigroup and it is necessarily a semilattice of groups. Jackson [31] proved that a non-Clifford inverse semigroup must be inherently non-dualisable. In this chapter, we initiate an investigation of dualisability within the class of Clifford semigroups that are semilattices of abelian groups.

6.1 Preliminaries

In this section we give an alternative definition of a Clifford semigroup and set up the notation that we require.

Definition 6.1.1. Let S be a nonempty set and let $\{\mathbf{G}_i \mid i \in S\}$ be a set of groups. Let $\mathbf{S} = \langle S; \wedge \rangle$ be a semilattice. Assume that for all $i \geq j$ in $\mathbf{S}$, there is a connecting homomorphism $u_{i,j} \colon \mathbf{G}_i \to \mathbf{G}_j$ such that

- $u_{i,i}$ is the identity map on $\mathbf{G}_i$,

- $u_{j,k} \circ u_{i,j} = u_{i,k}$, for all $i \geq j$ and $j \geq k$.

Let

$$M = \bigcup_{s \in S} (G_s \times \{s\})$$

and define $*\colon M^2 \to M$ by

$$(\forall i, j \in S)(\forall a \in G_i)(\forall b \in G_j)(a, i) * (b, j) := (u_{i,i\wedge j}(a) \cdot u_{j,i\wedge j}(b), i \wedge j).$$

Then $\mathbf{M} = \langle M; * \rangle$ is a *semilattice of groups.* It is clear that a semilattice of groups is a Clifford semigroup. By Howie (see [28, Page 107]), every Clifford semigroup is isomorphic to a semilattice of groups.

Notation 6.1.2. Let $\mathbf{M}$ be a semilattice of groups as introduced in Definition 6.1.1. Then, following Howie (see [28, Page 106]), we denote $\mathbf{M}$ by

$$\mathbf{M} = [\mathbf{S}; \mathbf{G}_s; u_{s,t}].$$

To simplify the notation, if $\mathbf{S} = \langle \{0, 1\}; \wedge \rangle$ with $0 < 1$ so that $\mathbf{M}$ is a semilattice of two groups, say $\mathbf{G}$ and $\mathbf{H}$, then we write $\mathbf{M} = [\mathbf{G}, u, \mathbf{H}]$ where u is the connecting homomorphism from $\mathbf{G}$ to $\mathbf{H}$. In general, we assume that the universe of $\mathbf{M}$ is $G \dot\cup H$ and unless it is specified we let $e^{(1)}, e^{(0)}$ denote the identity elements of $\mathbf{G}$ and $\mathbf{H}$, respectively. For example (see Figure 6.1), the familiar Clifford semigroups $\mathbf{G}^1$ and $\mathbf{G}^0$ are given as follows.

- $\mathbf{G}^1$ is isomorphic to $[\{1\}, u, \mathbf{G}]$, a Clifford semigroup on the universe $G \dot\cup \{1\}$ where the connecting homomorphism $u\colon 1 \mapsto e^{(0)}$ is the obvious homomorphism.

- $\mathbf{G}^0$ is isomorphic to $[\mathbf{G}, v, \{0\}]$, a Clifford semigroup on the universe $G \dot\cup \{0\}$ where v is again the obvious connecting homomorphism.

In general $\mathbf{G}^1$ and $\mathbf{G}^0$ are group adjoined with new identity and a group adjoined with a zero, respectively.

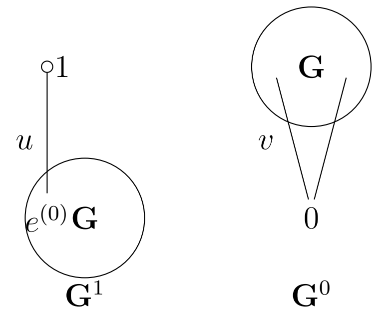

Figure 6.1: Comparison of $\mathbf{G}^1$ and $\mathbf{G}^0$

6.2 Duality of some semilattices of abelian groups

We divide this section into three subsections. We first investigate the dualisability of a cyclic group of order m adjoined with a new identity and then the dualisability of a dualisable group adjoined with a zero. Finally, we investigate the dualisability of a semilattice of cyclic groups of coprime orders.

6.2.1 Duality for a cyclic group of order m adjoined with an identity

In this subsection, we will prove that a cyclic group $\mathbf{C}_m$ of order m adjoined with a new identity is dualisable (Theorem 6.2.1), recalling that $\mathbf{C}_m$ is dualisable. We then use the fact to prove that every finite semilattice of abelian groups with injective connecting homomorphisms is dualisable. In general, it is not true that a dualisable semigroup adjoined with an identity need be dualisable. For example, Jackson [31] has shown that adjoining an identity to a finite non-trivial null semigroup gives a non-dualisable semigroup, however a null semigroup is dualisable [3]. For more examples see Lemma 2.5.2 and Jackson [31]. Let $\mathbf{C}_m = \langle C_m; * \rangle$ be a cyclic group of order m where e is the identity element, which is dualised by $\underset{\sim}{\mathbf{C}}_m = \langle C_m; *, \mathcal{T}\rangle$. Let $\mathbf{C}^1_m$ be $\mathbf{C}_m$ adjoined with 1 as a new identity element. Let r be the following ternary algebraic relation on $\mathbf{C}^1_m$:

$$r := C_m \times C^1_m \times C^1_m \cup \{(1, g, g) \mid g \in C_m\}.$$

Lemma 6.2.1. *The alter ego* $\underset{\sim}{\mathbf{C}}^1_m = \langle C^1_m; *, e, 1, r, \mathcal{T}\rangle$ *of* $\mathbf{C}^1_m$ *satisfies* (IC) *and consequently,* $\underset{\sim}{\mathbf{C}}^1_m$ *dualises* $\mathbf{C}^1_m$.

Proof. Let $\mathbf{X} \leqslant (\underset{\sim}{\mathbf{C}}^1_m)^n$ for some natural number n. Consider a morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{C}}^1_m$. Then, for all $x \in X$, we have $x_i^m \in \{1, e\}$. Let $B = \varphi^{-1}(\{1\})$. Then the set B is non-empty as the constant tuple $\underline{1} \in B$. Let $b^{(1)}, \dots, b^{(l)}$ be an enumeration of the elements of B. Define $\hat{1}$ to be the product of all elements of B to power m, that is,

$$\hat{1} := (b^{(1)} * \cdots * b^{(l)})^m.$$

It is clear that $\hat{1} \in B$. Let

$$I := \{i \le n \mid \hat{1}_i = 1\}.$$

We will show that $I \neq \varnothing$. Suppose that $I = \varnothing$; then for all $i \le n$, we have $\hat{1}_i \neq 1$. Thus $\hat{1}$ will be the constant tuple $\underline{e}$, which contradicts $\varphi(\hat{1}) = 1$. It now follows easily that if $\varphi(x) = 1$ then $x_i = 1$ for every $i \in I$. Conversely, let $x \in X$ be such that $x_i = 1$ for every $i \in I$. Then

$$x^m * \hat{1} = \hat{1} \Rightarrow \varphi(x^m) * \varphi(\hat{1}) = \varphi(\hat{1}) = 1 \Leftrightarrow (\varphi(x))^m = 1.$$

Hence

$$\varphi(x) = 1 \text{ if and only if for all } i \in I \text{ we have } x_i = 1. \qquad (\ddagger)$$

Equivalently, $\varphi(x) \in C_m$ if and only if there exists $i \in I$ such that $x_i \in C_m$.

Now, we show that I is a support for φ. Let $x, y \in X$ with $x\restriction_I = y\restriction_I$. By the earlier argument (‡), we may assume $\varphi(x), \varphi(y) \in C_m$. So

$$\varphi(x) = \varphi(x) * e = \varphi(x * \underline{e}) \text{ and } \varphi(y) = \varphi(y) * e = \varphi(y * \underline{e}).$$

(Now $(\hat{1}, x * \underline{e}, y * \underline{e}) \in r$.) We want to prove that $\varphi(x) = \varphi(y)$. Now

$$\begin{aligned}(\hat{1}, x * \underline{e}, y * \underline{e}) \in r &\Rightarrow (\varphi(\hat{1}), \varphi(x * \underline{e}), \varphi(y * \underline{e})) \in r^{\underset{\sim}{\mathbf{C}}^1_m} \\ &\Rightarrow (1, \varphi(x), \varphi(y)) \in r^{\underset{\sim}{\mathbf{C}}^1_m} \\ &\Rightarrow \varphi(x) = \varphi(y).\end{aligned}$$

Let $\pi_I \colon (C^1_m)^n \to (C^1_m)^I$ be the restriction to I. Let $T := \pi_I(X) \cap C^I_m$, a substructure of $\underset{\sim}{\mathbf{C}}^I_m$. Note that $\underset{\sim}{\mathbf{C}}_m = \langle C_m; *, \mathcal{T}\rangle$. Define $\psi \colon T \to C_m$ as follows: for all $z \in T$, let $\psi(z) := \varphi(x)$, where $x \in X$ and $z = \pi_I(x)$. Since I is a support for φ, it follows that ψ is well-defined as a function; moreover $\psi(T) \subseteq C_m$ because $\varphi(x) = 1$ if and only if $x\restriction_I = \underline{1}$. Because I is a support for φ and $*$ is total operation on $\mathbf{X}$, we have ψ preserves $*$ also. So, ψ is a morphism in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_m)$.

As $\underset{\sim}{\mathbf{C}}_m$ satisfies (IC), the morphism ψ agrees with some I-ary term function t; say $t = v_{i_1} * \cdots * v_{i_l}$ where $i_1, \ldots, i_l \in I$. Define the term s to be

$$s = v_{i_1} * \cdots * v_{i_l} * v^m_{i_{l+1}} * \cdots * v^m_{i_k}$$

where $\{i_1, \ldots, i_k\} = I$. We will show that φ extends to the term function s on X. If $x \in B$, then $\varphi(x) = 1 = s(x)$ as $x_i = 1$ for all $i \in I$. Let $x \in X \backslash B$. Then there exists $i \in I$ with $x_i \in C_m$. Hence $s(x) \in C_m$ and therefore

$$\begin{aligned}\varphi(x) = \varphi(x) * e = \varphi(x * \underline{e}) = \psi(\pi_I(x * \underline{e})) &= t(\pi_I(x * \underline{e})) \\ &= s(x * \underline{e}) = s(x) * e = s(x).\end{aligned}$$

Thus $\underset{\sim}{\mathbf{C}}^1_m$ satisfies (IC). Since $\underset{\sim}{\mathbf{C}}^1_m$ is of finite type, it follows that $\underset{\sim}{\mathbf{C}}^1_m$ dualises $\mathbf{C}^1_m$ by the (IC) Duality Theorem 2.4.3. □

We now show that any semilattice of groups with injective connecting homomorphisms with $\mathbf{G}$ as the bottom group, generates the same quasi-variety as $\mathbf{G}^1$.

Theorem 6.2.2. *Let $\mathbf{H} = [\mathbf{S}; \mathbf{G}_s; u_{s,t}]$ be a semilattice of groups with $|S| \geq 2$, assume that all connecting homomorphisms $u_{s,t}$, for $s \geq t$, are injective, and let the bottom group be $\mathbf{G}$. Then the semigroups $\mathbf{H}$ and $\mathbf{G}^1$ generate the same quasi-variety.*

Proof. Fix any $k \in S\backslash\{0\}$ and define $\alpha\colon \mathbf{G}^1 \to \mathbf{H}$ by

$$\alpha(x) = \begin{cases} (x, 0) & \text{if } x \in G, \\ (e_{\mathbf{G}_k}, k) & \text{if } x = 1. \end{cases}$$

Then α embeds $\mathbf{G}^1$ into $\mathbf{H}$ and hence, $\mathbf{G}^1 \in \mathbb{ISP}(\mathbf{H})$. Let $\beta\colon \mathbf{H} \to \mathbf{G}^1$ be a homomorphism defined by $\beta((g, i)) = u_{i,0}(g)$. Then β separates the elements of G_s, for each $s \in S$. For $i \in S$, define $\gamma_i\colon \mathbf{H} \to \mathbf{G}^1$ by

$$\gamma_i((g, k)) = \begin{cases} 1 & \text{if } i \leq k, \\ e & \text{if } i \nleq k. \end{cases}$$

If $x \in G_i$ and $y \in G_j$ with $i \neq j$, then either γ_i or γ_j will separate x and y. Hence, the homomorphisms $\{\gamma_i \mid i \in S\} \cup \{\beta\}$ will separate the elements of $\mathbf{H}$ into $\mathbf{G}^1$. Hence, $\mathbf{H} \in \mathbb{ISP}(\mathbf{G}^1)$. □

By combining Lemma 6.2.1, Theorem 6.2.2, Example 2.2.2 and recalling Theorem 5.1.1, we get the following corollary.

Theorem 6.2.3. *Every finite semilattice of abelian groups with injective connecting homomorphisms is dualisable.*

Lemma 6.2.4. *Let* $\mathbf{G}$ *be a finite abelian group. Then* $\mathbf{M} \in \mathbb{ISP}(\mathbf{G}^1)$ *if and only if* $\mathbf{M}$ *is isomorphic to a semilattice of abelian groups from the quasi-variety* $\mathbb{ISP}(\mathbf{G})$ *where the connecting homomorphisms are injective.*

Proof. Assume that $\mathbf{M}$ is a semilattice of groups from the quasi-variety $\mathbb{ISP}(\mathbf{G})$ with injective connecting homomorphisms and $\mathbf{B}$ as the bottom group. Since the group $\mathbf{B}$ is in the quasi-variety of $\mathbf{G}$, it follows that $\mathbf{B}^1 \in \mathbb{ISP}(\mathbf{G}^1)$. By Theorem 6.2.2, we have $\mathbf{M} \in \mathbb{ISP}(\mathbf{B}^1)$, hence $\mathbf{M} \in \mathbb{ISP}(\mathbf{G}^1)$.

Assume that $\mathbf{M} \in \mathbb{ISP}(\mathbf{G}^1)$. It is clear that $\mathbf{M}$ is a semilattice of abelian groups from the quasi-variety $\mathbb{ISP}(\mathbf{G})$, say that up to isomorphism, $\mathbf{M}$ is of the form $[\mathbf{S}; \mathbf{G}_s; u_{s,t}]$ where $|S| \geq 2$. Suppose by way of contradiction that there exists a connecting homomorphism $u_{i,j}$, for $i, j \in S$, which is not injective; so that there are $x, y \in G_i$ with $x \neq y$ and $u_{i,j}(x) = u_{i,j}(y)$. We want to show that x and y are not separated into $\mathbf{G}^1$.

Now, any homomorphism $\alpha \colon \mathbf{M} \to \mathbf{G}^1$ has either $\alpha(G_i) \subseteq \{1\}$ or $\alpha(G_i) \subseteq G$. If $\alpha(G_i) \subseteq \{1\}$ then x and y cannot be separated by α. If $\alpha(G_i) \subseteq G$ then as $j \leq i$, we have $\alpha(G_j) \subseteq G$ from which it easily follows that $\alpha(x) = \alpha(u_{i,j}(x)) = \alpha(u_{i,j}(y)) = \alpha(y)$; again x and y cannot be separated by α. Therefore, the Algebraic Separation Theorem 2.2.1 fails, contradicting $\mathbf{M} \in \mathbb{ISP}(\mathbf{G}^1)$. Hence, all connecting homomorphisms in $\mathbf{M}$ are injective. □

Example 6.2.5. *Let* $\mathbf{N} = [\mathbf{G}, u_1, \mathbf{G}]$ *be a semilattice of groups where* $\mathbf{G}$ *is a finite abelian group and the connecting homomorphism* u_1 *is given by* $u_1((g, 1)) = (g, 0)$*. Then* $\mathbf{G}^1$ *and* $\mathbf{N}$ *generate the same quasi-variety.*

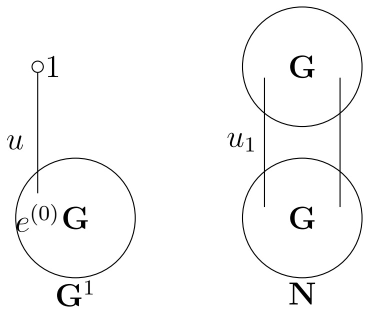

Figure 6.2: Comparison of $\mathbf{N}$ and $\mathbf{G}^1$

6.2.2 Duality for a dualisable group adjoined with zero

In this subsection, we will discuss the dualisability of a group adjoined by zero and some Clifford semigroups that generate the same quasi-variety. Let $\mathbf{M} = [\mathbf{G}, v_1, \mathbf{G}]$ be a semilattice of groups where $\mathbf{G}$ is a finite group. Let the connecting homomorphism $v_1 \colon g \mapsto e^{(0)}$ be the constant map. We aim to prove that $\mathbf{M}$ is dualisable depending on Davey and Knox [13] result presented in the following Example 6.2.6. Throughout the remainder of this chapter, we let $\mathbf{C}_n$ denote a cyclic group of order n.

Example 6.2.6. *Let $\mathbf{A}$ be a finite group dualisable by $\underset{\sim}{\mathbf{A}} = \langle A; G, H, R, \mathcal{T}\rangle$. Let $\mathbf{A}^0$ be the group $\mathbf{A}$ adjoined by zero as defined earlier in Notation 6.1.2. Then, by Davey and Knox* [13], *the alter ego*

$$\underset{\sim}{\mathbf{A}}^0 = \langle A^0; G^{A^0}, H^{A^0}, R^{A^0} \cup \{A\}, \mathcal{T}\rangle$$

dualises $\mathbf{A}^0$ where for each n-ary $g \in G$, define $g^{A^0} \colon (\mathbf{A}^0)^n \to \mathbf{A}^0$ by

$$g^{A^0}(x_1, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{if } x_1, \ldots, x_n \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x_1, \ldots, x_n \in A^0$. If $g \in G$ is nullary, then we define $g^{A^0} := g$. For *each n-ary $h \in H$, let* $\mathrm{dom}(h^{A^0}) = \mathrm{dom}(h) \dot\cup \{\underline{0}\}$ *where* $\underline{0}$ *is the n-tuple constant. Define* $h^{A^0} \colon \mathrm{dom}(h^{A^0}) \to \mathbf{A}^0$ *by*

$$h^{A^0}(x_1, \dots, x_n) = \begin{cases} h(x_1, \dots, x_n) & \textit{if } x_1, \dots, x_n \in A, \\ 0 & \textit{otherwise,} \end{cases}$$

for all $(x_1, \dots, x_n) \in \mathrm{dom}(h^{A^0})$*. For each n-ary $r \in R$, let*

$$r^{A^0} = r \dot\cup \{\underline{0}\}.$$

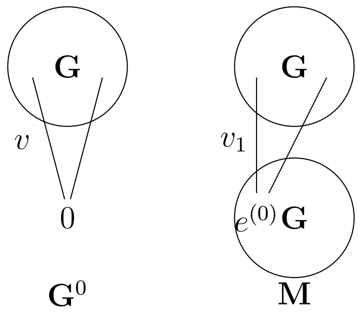

Figure 6.3: Comparison of $\mathbf{G}^0$ and $\mathbf{M}$

Theorem 6.2.7. *Let* $\mathbf{M} = [\mathbf{G}, v_1, \mathbf{G}]$ *be a semilattice of groups where* $\mathbf{G}$ *is a finite group. Let the connecting one-to-one homomorphism* $v_1 \colon g \mapsto e^{(0)}$ *be the constant map. Let* $\mathbf{G}^0$ *be the group* $\mathbf{G}$ *adjoined with* 0*. (See Figure 6.3.) Then* $\mathbf{G}^0$ *and* $\mathbf{M}$ *generate the same quasi-variety.*

Proof. It is clear that $\mathbf{G}^0 \in \mathbb{ISP}(\mathbf{M})$ since there is the one-to-one homomorphism $\beta \colon \mathbf{G}^0 \to \mathbf{M}$ defined by

$$\beta(x) = \begin{cases} (e^{(0)}, 0) & \text{if } x = 0, \\ (x, 1) & \text{otherwise.} \end{cases}$$

Define the homomorphism $\alpha_1\colon \mathbf{M} \to \mathbf{G}^0$ by

$$\alpha_1((x,i)) = \begin{cases} x & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, α_1 separates elements of $\mathbf{G}_1$. Define the map $\alpha_0\colon \mathbf{M} \to \mathbf{G}^0$ by

$$\alpha_0((x,i)) = \begin{cases} x & \text{if } i = 0, \\ e & \text{otherwise.} \end{cases}$$

Then the homomorphism α_0 separates elements of $\mathbf{G}_0$. Define the homomorphism $\gamma\colon \mathbf{M} \to \mathbf{G}^0$ by

$$\gamma((x,i)) = \begin{cases} e & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the homomorphism γ separates elements of G_0 from elements of G_1. Hence, $\{\alpha_0, \alpha_1, \gamma\}$ separates the elements of $\mathbf{M}$ into $\mathbf{G}^0$ and so $\mathbf{M} \in \mathbb{ISP}(\mathbf{G}^0)$. Therefore $\mathbf{M}$ and $\mathbf{G}^0$ generate the same quasi-variety. □

Combining Example 6.2.6 and Theorem 6.2.7, and recalling Theorem 2.4.4, we obtain the following theorem.

Theorem 6.2.8. *Let* $\mathbf{G}$ *be a finite dualisable group. Then the semigroup* $\mathbf{M} = [\mathbf{G}, v_1, \mathbf{G}]$ *is dualisable when* v_1 *is the constant map.*

6.2.3 Duality for a semilattice of cyclic groups of coprime orders

Now we will prove a two-element semilattice of cyclic groups with coprime orders is dualisable. We consider an example to demonstrate that such Clifford semigroups generate quasi-varieties not previously considered here.

Let $\mathbf{M}_1 = [\mathbf{C}_2, v_1, \mathbf{C}_3]$ be the semilattice of groups $\mathbf{C}_2, \mathbf{C}_3$ where $\mathbf{C}_2$ and $\mathbf{C}_3$ are cyclic groups of order 2 and 3, respectively; by necessity, the connecting homomorphism is the constant map $v_1 : g \mapsto e^{(0)}$.

Lemma 6.2.9. *Let* $\mathbf{G}$ *be a finite group, then* $\mathbf{G}^0$ *and* $\mathbf{M}_1 = [\mathbf{C}_2, v_1, \mathbf{C}_3]$ *do not generate the same quasi-variety.*

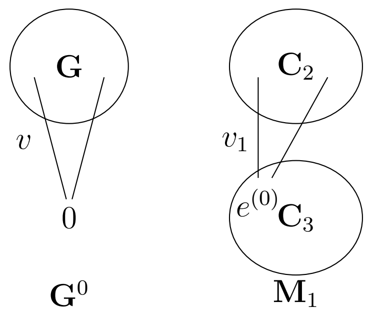

Figure 6.4: Comparison of $\mathbf{G}^0$ and $\mathbf{M}_1$

Proof. Suppose by way of contradiction that $\mathbf{G}^0$ and $\mathbf{M}_1$ generate the same quasi-variety, then $\mathbf{G}$ must contain a group element of order 2 and a group element of order 3. Let $g \in G$ be the group element of order 3. Since $\mathbf{G}^0 \in \mathbb{ISP}(\mathbf{M}_1)$, we choose $\beta : \mathbf{G}^0 \to \mathbf{M}_1$ such that $\beta(e) \neq \beta(g)$. It follows that $\beta(0), \beta(e), \beta(g) \in C_3$. Therefore,

$$\beta(g) * \beta(0) = \beta(g * 0) = \beta(0) = \beta(e * 0) = \beta(e) * \beta(0).$$

By right cancellation, we have $\beta(g) = \beta(e)$, a contradiction. □

Lemma 6.2.10. *Let* $\mathbf{M}_1 = [\mathbf{C}_2, v_1, \mathbf{C}_3]$ *be the semilattice of cyclic groups* $\mathbf{C}_2, \mathbf{C}_3$*, as defined above. Let* $\mathbf{G}^1$ *be a finite group adjoined with a new identity as defined in Notation 6.1.2. Then* $\mathbf{M}_1$ *and* $\mathbf{G}^1$ *do not generate the same quasi-variety.*

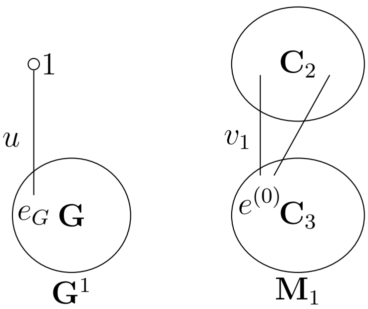

Figure 6.5: Comparison of $\mathbf{G}^1$ and $\mathbf{M}_1$

Proof. Suppose there is a homomorphism $\alpha\colon \mathbf{M}_1 \to \mathbf{G}^1$ that separates elements of C_2 with $\alpha(e^{(1)}) \neq \alpha(a)$ and $a \in C_2$. Since a and $e^{(1)}$ lie in the same subgroup and $\alpha(a)*\alpha(e^{(1)}) = \alpha(a) \in G$, it follows that $\alpha(a), \alpha(e^{(1)}) \in G$. We claim that $\alpha(e^{(0)}) \in G$: suppose, by way of contradiction that, $\alpha(e^{(0)}) = 1$, then

$$1 = \alpha(e^{(0)}) = \alpha(e^{(0)} * a) = \alpha(e^{(0)}) * \alpha(a) \in G,$$

a contradiction. It follows that $\alpha(e^{(0)}) = \alpha(e^{(1)}) = e_G$. So,

$$\alpha(e^{(1)}) = \alpha(e^{(0)}) = \alpha(e^{(0)} * a) = \alpha(e^{(0)}) * \alpha(a) = e_G * \alpha(a) = \alpha(a),$$

a contradiction. Hence $\mathbf{M}_1$ and $\mathbf{G}^1$ do not generate the same quasivariety. □

We will prove a two-element semilattice of cyclic groups of coprime orders is dualisable. Let $\mathbf{D} = [\mathbf{C}_q, v, \mathbf{C}_p]$ be the semilattice of groups $\mathbf{C}_q, \mathbf{C}_p$, where $p, q \in \mathbb{N}$ with p and q coprime and $v\colon g \mapsto e^{(0)}$ is the only connecting homomorphism. Let $\underset{\sim}{\mathbf{C}}_q = \langle C_q; *, \mathcal{T}\rangle$ and $\underset{\sim}{\mathbf{C}}_p = \langle C_p; *, \mathcal{T}\rangle$ be the alter egos of $\mathbf{C}_q = \langle C_q, *\rangle$ and $\mathbf{C}_p = \langle C_p, *\rangle$, respectively. Let r be the following ternary relation:

$$r := C_p \times D \times D \cup \{(e^{(1)}, g, g) \mid g \in D\},$$

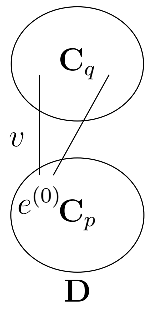

Figure 6.6: Illustration of **D**

where $e^{(1)}$ is the identity of $\mathbf{C}_q$. We will show that the relation r is algebraic by considering only the following non-trivial case: let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in r$ with $x_1 = x_2 = e^{(1)}$. Then we have

$$\begin{aligned}(x_1, y_1, z_1) * (x_2, y_2, z_2) &= (x_1 * x_2, y_1 * y_2, z_1 * z_2) \\ &= (e^{(1)}, y_1 * y_2, z_1 * z_2) \qquad \text{as } x_1 = x_2 = e^{(1)} \\ &\Rightarrow (e^{(1)}, y_1 * y_2, z_1 * z_2) \in r \quad \text{as } y_1 = z_1,\ y_2 = z_2 \\ &\Rightarrow (x_1, y_1, z_1) * (x_2, y_2, z_2) \in r.\end{aligned}$$

Hence, the relation r is algebraic.

Theorem 6.2.11. *The alter ego* $\underset{\sim}{\mathbf{D}} = \langle D; *, e^{(0)}, e^{(1)}, r, \mathcal{T}\rangle$ *dualises* $\mathbf{D}$.

Proof. Let $\mathbf{X} \leqslant \underset{\sim}{\mathbf{D}}^n$ for some natural number n. Let $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ be a morphism. Let $A = \varphi^{-1}(C_q)$. Since $\varphi(\underline{e}^{(1)}) = e^{(1)}$ then $A \neq \varnothing$. Define

$$\widehat{x} := \prod_{x \in A} x^{pq}.$$

It is clear that $\widehat{x} \in A$ and $\varphi(\widehat{x}) = e^{(1)}$. Let $I := \{i \leq n \mid \widehat{x}_i = e^{(1)}\}$. Since the constant tuple $\underline{e}^{(0)} \notin A$ then $I \neq \varnothing$. We claim that $x \in A$ if and only if $x_i \in C_q$ for all $i \in I$. Now if $x \in A$ then for all $i \in I$, we have $x_i \in C_q$ by definition of $\widehat{x}$ and the set I. Conversely, if $x_i \in C_q$ for all $i \in I$ then

$x^{pq} * \widehat{x} = \widehat{x}$ and $\varphi(x^{pq}) * \varphi(\widehat{x}) = \varphi(\widehat{x}) = e^{(1)}$. Therefore $\varphi(x)^{pq} = e^{(1)}$ and hence $x \in A$. Equivalently, $\varphi(x) \in C_p$ if and only if there exists $i \in I$ such that $x_i \in C_p$.

Let $\pi_I \colon D^n \to D^I$ be the restriction to I. We claim that I is a support set for φ. Let $x, y \in X$ with $\pi_I(x) = \pi_I(y)$. We will show that $\varphi(x) = \varphi(y)$. We have

$$\begin{aligned} &(\widehat{x}, x, y) \in r \\ \Rightarrow &(\varphi(\widehat{x}), \varphi(x), \varphi(y)) \in r^{\underset{\sim}{\mathbf{D}}} \\ \Rightarrow &(e^{(1)}, \varphi(x), \varphi(y)) \in r^{\underset{\sim}{\mathbf{D}}}, \end{aligned}$$

and hence $\varphi(x) = \varphi(y)$. Since $\pi_I(A) \subseteq C_q^I$ and $\underset{\sim}{\mathbf{C}}_q = \langle C_q, *, \mathfrak{T}\rangle$, it follows that $\pi_I(A)$ is a substructure of $\underset{\sim}{\mathbf{C}}_q^I$. Define $\psi_1 \colon \pi_I(A) \to C_q$ as follows: for all $z \in \pi_I(A)$, let $\psi_1(z) := \varphi(x)$, where $x \in A$ and $\pi_I(x) = z$. Since I is a support set for φ then ψ_1 is well-defined function. Because I is a support for φ and $*$ is total on $\mathbf{X}$, we have ψ_1 is a morphism in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_q)$.

As $\underset{\sim}{\mathbf{C}}_q$ satisfies (IC), therefore ψ_1 agrees with some I-ary term function t_1; say $t_1 = v_{i_1}^{m_1} * \dots * v_{i_l}^{m_l}$ where $\{i_1, \dots, i_l\} = I$ and $m_1, \dots, m_l \in \{1, \dots, q\}$.

Let

$$U = X \backslash \{x \in X \mid (\exists i \in I) \quad x_i \in C_q\}$$

and $T = \pi_I(U)$. Note that $T = \pi_I(\underline{e}^{(0)} * X)$. Clearly, $\mathbf{T}$ is a substructure of $\underset{\sim}{\mathbf{C}}_p^I$. Define $\psi_2 \colon T \to C_p$ as follows: for $z \in T$, let $\psi_2(z) := \varphi(x)$, where $x \in X$ and $\pi_I(x) = z$. Since I is a support set for φ then ψ_2 is a well-defined function. Because I is a support for φ and $*$ is total on $\mathbf{X}$, we have ψ_2 is a morphism in $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{C}}_p)$.

Since $\underset{\sim}{\mathbf{C}}_p$ satisfies (IC), it follows that ψ_2 extends to some term function $t_2 = v_{i_1}^{n_1} * \dots * v_{i_l}^{n_l}$ where $n_1, \dots, n_l \in \{1, \dots, p\}$. We claim that for each $i \in I$, there exists $d_i \in \{1, \dots, pq\}$ such that φ extends to some term function

$$s(x) = \prod_{i \in I} x_i^{d_i}.$$

If $x_i^{m_i}$ appears in t_1 and $x_i^{n_i}$ in t_2 then by Chinese Remainder Theorem there exists $d_i \leq pq$ such that $m_i \equiv d_i \pmod{q}$, $n_i \equiv d_i \pmod{p}$. Hence the variable $x_i^{d_i}$ is in term s. Now let $x \in A$. Then

$$\varphi(x) = \psi_1(\pi_I(x)) = t_1(\pi_I(x)) = s(x).$$

Let $x \in X \backslash A$. Then, $\varphi(x) \in C_p$ and so

$$\begin{aligned}\varphi(x) = \varphi(x) * e^{(0)} &= \varphi(x * \underline{e}^{(0)}) \\ &= \psi_2(\pi_I(x * \underline{e}^{(0)})) \\ &= t_2(\pi_I(x * \underline{e}^{(0)})) = s(x * \underline{e}^{(0)}) = s(x) * e^{(0)} = s(x).\end{aligned}$$

Hence for all $x \in X$, we have $\varphi(x) = s(x)$. Therefore, $\underset{\sim}{\mathbf{D}}$ satisfies (IC) and dualises $\mathbf{D}$. $\square$

Conclusion

In this thesis, we have investigated the dualisability problem for some quasi-varieties of semigroups, in particular, the quasi-varieties of normal bands, some completely simple semigroups and some Clifford semigroups.

We completed the characterization of dualisability for finite bands in Chapter 3. We showed that all remaining quasi-varieties of normal bands admit a natural duality. We concluded the chapter by showing that the direct product of a dualisable algebra with a 2-element right-zero semigroup with constant can be inherently non-dualisable.

In general, the direct product of two dualisable algebras need not be dualisable. In Chapter 4, we examined the special case of independent varieties and showed that in this case, the direct product of two dualisable algebras is indeed dualisable. The results in this chapter provided one half of the likely classification of dualisability for completely simple semigroups. Extending the corresponding results to the level of the fully dualisable and strongly dualisable algebras has not been studied in this thesis and will be considered in further work.

In Chapter 5, motivated by some issues that arise in Chapter 6, we proved the (IC) Transfer Theorem 5.1.1. It says that if two algebras generate the same quasi-variety and one of these algebras has an alter ego of finite type that satisfies the Interpolation Condition (IC), then the alter ego of other algebra does as well.

In Chapter 6, we investigated dualisability within the class of Clifford semigroups that are semilattices of abelian groups. We found dualities for a cyclic group of order m adjoined with a new identity and gave some Clifford semigroups that generate the same quasi-varieties as this semigroup. We established the dualisability of any finite semilattice of abelian groups in which the connecting homomorphisms are injective. Finally, we found a duality for each semilattice of two cyclic groups of coprime orders. There are other Clifford semigroups that have not been investigated for dualisability; for example, an arbitrary dualisable group adjoined with a new identity and semilattices of dualisable groups when the connecting homomorphisms are non-trivial and non-injective. It would be interesting to continue further the investigation for dualisability of these examples, which may lead us to general results.

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