Compatible relations on logic-based algebras

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Contents

Summary

This thesis is devoted to the study of finite algebras that admit only finitely many compatible relations, up to a natural interdefinability. The study of these algebras was originally motivated by the theory of natural dualities. The general problem of determining which finite algebras admit only finitely many relations seems difficult. As this finiteness condition implies the existence of a near-unanimity term, it is natural to investigate the condition among lattice-based algebras. In this thesis, we focus on two familiar varieties of algebras related to non-classical logics:

- building on the work of Davey and Pitkethly, we complete the characterisation of finite Heyting algebras admitting only finitely many compatible relations, and
- we characterize finite Ockham algebras admitting only finitely many compatible relations.

The techniques used for Heyting algebras and for Ockham algebras are quite different.

We give a general sufficient condition for a finite algebra to admit only finitely many compatible relations. This condition is closely related to endoprimality. We verify that finite Heyting chains satisfy this condition, and thereby complete the proof that a finite Heyting algebra admits only finitely many compatible relations if and only if it is a chain.

For Ockham algebras, we use two different duality techniques: the restricted Priestley duality for Ockham algebras and the piggyback duality for the quasi-variety generated by a finite subdirectly irreducible Ockham algebra. The characterisation of finite Ockham algebras that admit only finitely many compatible relations is stated in terms of the restricted Priestley dual spaces, with the natural duality being used to represent compatible relations on certain finite subdirectly irreducible Ockham algebras. Our characterisation within the variety of Ockham algebras can be easily restricted to familiar subvarieties: for example, the varieties of Boolean algebras, Stone algebras, De Morgan algebras, Kleene algebras and MS algebras.

Statement of Authorship

Except where reference is made in this text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis submitted for the award of any other degree or diploma.

No other person's work has been used without due acknowledgement in the main text of the thesis.

The thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

Thanh Long NGUYEN **Melbourne,**....................................

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Introduction

Up to term equivalence, a finite algebra is determined by its compatible relations, so it is natural to study finite algebras by studying their compatible relations. The compatible relations on a finite algebra have been studied from a number of different perspectives. Many authors have investigated when the clone of a finite algebra is determined by a finite set of relations. The algebras satisfying this property are called *finitely related*. All finite algebras within the familiar classes of lattices [2], groups [1, 39, 45, 46] and commutative semigroups [21] are finitely related, as are finite semilattices and unary algebras.

In the study of constraint satisfaction problems, it is natural to look at how fast the number of n-ary compatible relations on a finite algebra grows. Berman, Idziak, Markovic, McKenzie, Valeriote and Willard [6] characterised the class of finite algebras with *few subpowers*. All finite lattices and groups have few subpowers, but non-trivial finite semilattices and unary algebras do not.

In this thesis, we study finite algebras that admit only finitely many compatible relations, up to a natural interdefinability. This finiteness condition, which arises naturally in duality theory, was introduced recently by Davey and Pitkethly [24]. It is closely related to, and stronger than, the classic finiteness condition given in 1975 by Baker and Pixley [2]:

- There is a finite set of compatible relations on a finite algebra $\mathbf{A}$ from which all other compatible relations on $\mathbf{A}$ can be defined, via conjunctions of atomic formulæ.

They proved that this holds if and only if $\mathbf{A}$ has a near-unanimity term.

A finite algebra **A** is said to *admit only finitely many relations* if there is a finite set of compatible relations on **A** such that every other compatible relation on **A** is interdefinable with one of these relations, via conjunctions of atomic formulæ. The general problem of determining ***which finite algebras admit only finitely many relations*** seems difficult.

Based on duality theory, Davey and Pitkethly [24] developed the fundamental techniques to make this problem easier to tackle. Amongst all the alter egos of a finite algebra **A**, they showed that an alter ego satisfying the interpolation condition (IC) plays a crucial role in encoding compatible relations on **A**. This condition guarantees that all compatible relations on **A** can be encoded using certain finite structures related to that alter ego. When encoded in this way, it can be much easier to show that two compatible relations are conjunct-atomic definable from each other.

Using these techniques, Davey and Pitkethly [24] obtained the following results:

- a finite Boolean algebra **B** admits only finitely many relations if and only if $|B| \leqslant 2$;
- a finite lattice **L** admits only finitely many relations if and only if $|L| \leqslant 2$;
- every finite Stone algebra with more than three elements admits infinitely many relations;
- a finite Heyting algebra that is not a chain admits infinitely many relations.

These results seem to suggest that not many finite algebras have this finiteness property.

In this thesis, we aim to find more finite algebras that satisfy this finiteness property. As this property implies the existence of a near-unanimity term, it is natural to study lattice-based algebras. (Note that every lattice-based algebra has a ternary near-unanimity term: the median term.) We focus on two familiar varieties of algebras related to non-classical logics: the variety of Heyting algebras and the variety of Ockham algebras. We give a complete description of the finite algebras within these two varieties that admit only finitely many relations. We develop a general sufficient condition for a finite algebra to admit only finitely many relations. We then use this condition to complete the characterisation for the variety of Heyting algebras. We use Priestley duality and piggyback duality to help us obtain the result within the variety of Ockham algebras.

To make use of the techniques mentioned above to encode compatible relations on a finite algebra, it is essential to find an alter ego satisfying (IC). Finding a manageable alter ego satisfying this condition can be difficult. In this thesis, for a particular family of Ockham algebras, we use piggyback techniques to find alter egos that yield a strong duality and so satisfy (IC). We then use these alter egos to encode the compatible relations on the original Ockham algebras.

In Chapter 1, we introduce the basic notions concerning counting the compatible relations on a finite algebra, the technique used to encode compatible relations and the results required for the remainder of this thesis. We also give a brief introduction to natural dualities and provide a fundamental example of a natural duality that we need later, namely, Priestley duality for bounded distributive lattices.

In Chapter 2, we introduce strictly p-endoprimal algebras. This algebraic property is closely related to endoprimality. We also generalise the property of being linear from unary algebras to partial unary algebras. Linearity ensures that the structures used to encode compatible relations are simply represented. We then give a general sufficient condition for a finite algebra to admit only finitely many relations. This condition is related to the two properties introduced above. Finally, we verify that finite Heyting chains satisfy this condition, and thereby complete the proof that a finite Heyting algebra admits only finitely many relations if and only if it is a chain. This chapter is based on the paper written with my supervisor J. G. Pitkethly [51].

In Chapter 3, we introduce two different duality techniques: the restricted Priestley duality for Ockham algebras and the piggyback duality for the quasivariety generated by a finite subdirectly irreducible Ockham algebra. These duality techniques will be used to encode compatible relations on finite Ockham algebras in Chapter 4. We finish Chapter 3 by investigating an example of a family of finite subdirectly irreducible Ockham algebras that play a crucial role in describing the finite Ockham algebras that admit only finitely many relations.

In Chapter 4, we use the two duality techniques introduced in Chapter 3 to complete the characterisation of finite Ockham algebras that admit only finitely many relations. The characterisation is stated in terms of Priestley dual spaces. Up to isomorphism and symmetry, the Ockham algebras that admit only finitely many relations can be grouped into two countably infinite families. The first family generalises the two-element Boolean algebra and the second family generalises the three-element Stone algebra. Our characterisation within the variety of Ockham algebras can be easily restricted to familiar subvarieties: for example, the varieties of Boolean algebras, Stone algebras, Kleene algebras, De Morgan algebras and MS algebras. This chapter is based on a paper written with my supervisors B. A. Davey and J. G. Pitkethly [22].

Chapter 1

Preliminaries

In this chapter, we give the basic definitions and results required for this thesis. In the first two sections, we introduce the fundamental definitions and results concerning counting compatible relations on a finite algebra. We begin in Section 1.1 by giving the definitions of a structure and some morphisms between structures. Most of the notions concerning structures are from the book by Clark and Davey [13]. In Section 1.2, we use certain structures related to a finite algebra to encode its compatible relations. This representation technique comes from the paper by Davey and Pitkethly [24] and is based on the theory of natural dualities. In the last section, we give a brief introduction to natural dualities and introduce a fundamental example of a natural duality, Priestley duality for bounded distributive lattices.

1.1 Structures

Consider the following sets of symbols: a set G of finitary total operation symbols, a set H of finitary partial operation symbols and a set R of finitary relation symbols. Each symbol in $G \cup H \cup R$ carries an arity. We allow total operations to be nullary, but partial operations and relations must have positive arities. Let X be a non-empty set. For $n \geqslant 0$, an n-ary total operation on X is a map from X^n to X. For $n \geqslant 1$, an n-ary partial operation on X is a map from a subset of X^n to X. For $n \geqslant 1$, an n-ary relation on X is a subset of X^n.

Definition 1.1.1. A **structure** $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}} \rangle$ of type $\langle G, H, R \rangle$ is a non-empty set X endowed with a collection

(i) $G^{\mathbb{X}}$ consisting of an n-ary total operation $g^{\mathbb{X}}$ on X for each n-ary total operation symbol $g \in G$;

(ii) $H^{\mathbb{X}}$ consisting of an n-ary partial operation $h^{\mathbb{X}}$ on X for each n-ary partial operation symbol $h \in H$;

(iii) $R^{\mathbb{X}}$ consisting of an n-ary relation $r^{\mathbb{X}}$ on X for each n-ary relation symbol $r \in R$.

If $R^{\mathbb{X}} = \varnothing$, we refer to $\mathbb{X}$ as a **partial algebra**. In the case $H^{\mathbb{X}} = \varnothing$ and $R^{\mathbb{X}} = \varnothing$, we refer to $\mathbb{X}$ as an **algebra**. If X is finite, then we call $\mathbb{X}$ a **finite structure**. If $G^{\mathbb{X}}$, $H^{\mathbb{X}}$ and $R^{\mathbb{X}}$ are finite then we say $\mathbb{X}$ is a structure of **finite type**. In this thesis, we shall deal mainly with finite structures of finite type. A **topological structure** of type $\langle G, H, R \rangle$ is a (possibly empty) structure of type $\langle G, H, R \rangle$ endowed with a topology.

Remark 1.1.2. In the definition of a structure, we do not allow the empty structure, but we allow the empty topological structure. This is because an empty topological structure can be needed as part of a duality.

As above, we shall use superscripts on G, H, R and their members to indicate which structure we are working in. Where there is no confusion, we shall omit the superscripts to simplify notation. In this thesis, we use blackboard bold symbols to denote structures. In the case where the structures are algebras, we shall use bold symbols to denote them.

Definition 1.1.3. Let $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}} \rangle$ and $\mathbb{Y} = \langle Y; G^{\mathbb{Y}}, H^{\mathbb{Y}}, R^{\mathbb{Y}} \rangle$ be two structures of the same type. The structure $\mathbb{Y}$ is called a **substructure** of the structure $\mathbb{X}$, written $\mathbb{Y} \leqslant \mathbb{X}$, provided that $Y \subseteq X$ and

(i) for each n-ary $h \in G \cup H$, we have $\operatorname{dom}(h^{\mathbb{Y}}) = \operatorname{dom}(h^{\mathbb{X}}) \cap Y^n$ and $h^{\mathbb{Y}}$ agrees with $h^{\mathbb{X}}$ on $\operatorname{dom}(h^{\mathbb{Y}})$, and

(ii) for each n-ary $r \in R$, we have $r^{\mathbb{Y}} = r^{\mathbb{X}} \cap Y^n$.

Definition 1.1.4. Let $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}} \rangle$ and $\mathbb{Y} = \langle Y; G^{\mathbb{Y}}, H^{\mathbb{Y}}, R^{\mathbb{Y}} \rangle$ be two structures of the same type. A map $\varphi \colon X \to Y$ is a **morphism**, written $\varphi \colon \mathbb{X} \to \mathbb{Y}$, if it preserves each member of $G \cup H \cup R$, that is,

(i) for each n-ary $g \in G$ and each $(x_1, x_2, \dots, x_n) \in X^n$, we have

$$\varphi(g^{\mathbb{X}}(x_1, x_2, \dots, x_n)) = g^{\mathbb{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(ii) for each n-ary $h \in H$ and each $(x_1, x_2, \dots, x_n) \in \mathrm{dom}(h^{\mathbb{X}})$, we have $(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in \mathrm{dom}(h^{\mathbb{Y}})$ and

$$\varphi(h^{\mathbb{X}}(x_1, x_2, \dots, x_n)) = h^{\mathbb{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(iii) for each n-ary $r \in R$ and each $(x_1, x_2, \dots, x_n) \in r^{\mathbb{X}}$, we have

$$(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in r^{\mathbb{Y}}.$$

A morphism between two algebras is normally called a **homomorphism**. A morphism from an algebra to itself is called an **endomorphism**. A **partial endomorphism** of an algebra $\mathbf{A}$ is a homomorphism from a subalgebra of $\mathbf{A}$ to $\mathbf{A}$. A morphism $\varphi \colon \mathbb{X} \to \mathbb{Y}$ is a **retraction** if there exists a morphism $\psi \colon \mathbb{Y} \to \mathbb{X}$ such that $\varphi \circ \psi = \mathrm{id}_Y$. The morphism ψ is then called a **co-retraction**. In most cases, we have $\mathbb{Y} \leqslant \mathbb{X}$ and $\varphi{\restriction}_Y = \mathrm{id}_Y$, and so the co-retraction ψ is the inclusion map. A morphism $\varphi \colon \mathbb{X} \to \mathbb{Y}$ is an **isomorphism** if φ is a bijection and φ^{-1} is a morphism. If $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic, we write $\mathbb{X} \cong \mathbb{Y}$. A morphism $\varphi \colon \mathbb{X} \to \mathbb{Y}$ is an **embedding** if $\varphi(\mathbb{X})$ forms a substructure of $\mathbb{Y}$ and φ is an isomorphism from $\mathbb{X}$ to $\varphi(\mathbb{X})$. In practice, we use the following equivalent condition for a morphism to be an embedding.

Lemma 1.1.5 ([13, 1.4.2])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be two structures of the same type $\langle G, H, R \rangle$. A morphism $\varphi \colon \mathbb{X} \to \mathbb{Y}$ is an embedding if and only if the following hold:*

(1) *φ is one-to-one,*

(2) *for each n-ary $r \in \mathrm{dom}(H) \cup R$ and for each $(x_1, x_2, \ldots, x_n) \in X^n$, whenever $(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)) \in r^{\mathbb{Y}}$, we have $(x_1, x_2, \ldots, x_n) \in r^{\mathbb{X}}$.*

In particular, a morphism $\varphi\colon \mathbb{X} \to \mathbb{Y}$ is an isomorphism if and only if it is bijective and satisfies (2)*.*

Let $\boldsymbol{\mathcal{X}}$ be a class of structures of the same type. In this thesis, we shall deal with the following classes:

- $\mathsf{I}(\boldsymbol{\mathcal{X}})$, the class of isomorphic copies of members of $\boldsymbol{\mathcal{X}}$;
- $\mathsf{S}(\boldsymbol{\mathcal{X}})$, the class of substructures of members of $\boldsymbol{\mathcal{X}}$;
- $\mathsf{H}(\boldsymbol{\mathcal{X}})$, the class of homomorphic images of members of $\boldsymbol{\mathcal{X}}$;
- $\mathsf{P}(\boldsymbol{\mathcal{X}})$, the class of direct products of members of $\boldsymbol{\mathcal{X}}$ (possibly over the empty index set);
- $\mathsf{P_f}(\boldsymbol{\mathcal{X}})$, the class of direct products of members of $\boldsymbol{\mathcal{X}}$ over a non-empty finite index set.

A **variety** is a class $\boldsymbol{\mathcal{V}}$ of algebras closed under the operators H, S and P. If the variety $\boldsymbol{\mathcal{V}} = \mathsf{HSP}(\mathbf{M})$, then we say $\boldsymbol{\mathcal{V}}$ is **generated** by $\mathbf{M}$. A **subvariety** is a subclass of $\boldsymbol{\mathcal{V}}$ that is itself a variety.

In this thesis, we shall give characterisations of the finite algebras that admit only finitely many relations within the variety of Heyting algebras and the variety of Ockham algebras. The general concepts and techniques for solving these problems are given in the following sections.

1.2 Compatible relations on a finite algebra

Given a set $\{v_1, v_2, \ldots, v_n\}$ of variables, an n-ary **term** $t = t(v_1, \ldots, v_n)$ of type $\langle G, H, R\rangle$ is a string of symbols and is defined recursively as follows:

(i) each variable and each nullary operation symbol of G is an n-ary term, and

(ii) if $t_1, t_2, \dots, t_k$ are n-ary terms and $h \in G \cup H$ is a k-ary partial operation symbol, then $h(t_1, \dots, t_k)$ is an n-ary term.

Let $\mathbb{X}$ be a structure of type $\langle G, H, R\rangle$ and assume that t is an n-ary term of the same type. Then t can be interpreted in the natural way as an n-ary partial operation $t^{\mathbb{X}}$ on $\mathbb{X}$, where $t^{\mathbb{X}}$ has the maximum possible domain. The n-ary partial operation $t^{\mathbb{X}}$ on $\mathbb{X}$ is called an n-ary **term function** of $\mathbb{X}$.

An n-ary **atomic formula** of type $\langle G, H, R\rangle$ is an expression of one of the forms

$$t_1 \approx t_2 \quad \text{and} \quad r(t_1, t_2, \dots, t_k)$$

where $t_1, t_2, \dots, t_k$ are n-ary terms of type $\langle G, H, R\rangle$ and $r \in R$ is k-ary. Let $\mathbb{X}$ be a structure of type $\langle G, H, R\rangle$. Assume that t_1 and t_2 are n-ary terms of the same type. Let $x_1, \dots, x_n \in X$. Then $\mathbb{X}$ satisfies $t_1^{\mathbb{X}}(x_1, \dots, x_n) = t_2^{\mathbb{X}}(x_1, \dots, x_n)$ if $(x_1, \dots, x_n) \in \mathrm{dom}(t_1^{\mathbb{X}}) \cap \mathrm{dom}(t_2^{\mathbb{X}})$ and the two sides are equal. Similarly, for n-ary terms $t_1, t_2, \dots, t_k$ and a k-ary $r \in R$, we say that the structure $\mathbb{X}$ satisfies $r^{\mathbb{X}}(t_1^{\mathbb{X}}(x_1, \dots, x_n), \dots, t_k^{\mathbb{X}}(x_1, \dots, x_n))$ if $(x_1, x_2, \dots, x_n) \in \mathrm{dom}(t_i^{\mathbb{X}})$, for $i \in \{1, \dots, k\}$ and $\big(t_1^{\mathbb{X}}(x_1, \dots, x_n), \dots, t_k^{\mathbb{X}}(x_1, \dots, x_n)\big) \in r^{\mathbb{X}}$. We shall write $\mathbb{X} \models \Phi(x_1, \dots, x_n)$ to denote that $\mathbb{X}$ satisfies the atomic formula $\Phi(v_1, \dots, v_n)$ at $(x_1, \dots, x_n)$.

Definition 1.2.1. Let $\mathbb{X}$ be a structure of type $\langle G, H, R\rangle$. For each $n \geqslant 1$, we say that an n-ary relation r on X is **conjunct-atomic definable from** $\mathbb{X}$ if there exists a finite index set I and a set $\{\, \Phi_i(v_1, \dots, v_n) \mid i \in I \,\}$ of atomic formulæ of type $\langle G, H, R\rangle$ such that

$$r = \{\, (x_1, \dots, x_n) \in X^n \mid (\forall i \in I)\ \mathbb{X} \models \Phi_i(x_1, \dots, x_n) \,\}.$$

Alternatively, we say that r is conjunct-atomic definable from $G^{\mathbb{X}} \cup H^{\mathbb{X}} \cup R^{\mathbb{X}}$.

Example 1.2.2. 1. Let $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}}\rangle$ be a finite structure. Then the following relations are conjunct-atomic definable from $\mathbb{X}$:

(i) The domain of an n-ary partial operation $h^{\mathbb{X}} \in G^{\mathbb{X}} \cup H^{\mathbb{X}}$,

$$\mathrm{dom}(h^{\mathbb{X}}) = \{(x_1, \dots, x_n) \in X^n \mid h^{\mathbb{X}}(x_1, \dots, x_n) = h^{\mathbb{X}}(x_1, \dots, x_n)\}.$$

(ii) The graph of an n-ary partial operation $h^{\mathbb{X}} \in G^{\mathbb{X}} \cup H^{\mathbb{X}}$,

$$\mathrm{graph}(h^{\mathbb{X}}) = \{(x_1, x_2, \dots, x_n, y) \in X^{n+1} \mid h^{\mathbb{X}}(x_1, x_2, \dots, x_n) = y\}.$$

(iii) The equaliser of two n-ary partial operations $h_1^{\mathbb{X}}, h_2^{\mathbb{X}} \in G^{\mathbb{X}} \cup H^{\mathbb{X}}$,

$$\mathrm{eq}(h_1^{\mathbb{X}}, h_2^{\mathbb{X}}) = \{(x_1, \dots, x_n) \in X^n \mid h_1^{\mathbb{X}}(x_1, \dots, x_n) = h_2^{\mathbb{X}}(x_1, \dots, x_n)\}.$$

(iv) The fix-point set of a unary partial operation $h^{\mathbb{X}}$ in $G^{\mathbb{X}} \cup H^{\mathbb{X}}$,

$$\mathrm{fix}(h^{\mathbb{X}}) = \{x \in X \mid h^{\mathbb{X}}(x) = x\}.$$

(v) The kernel of a unary partial operation $h^{\mathbb{X}} \in G^{\mathbb{X}} \cup H^{\mathbb{X}}$,

$$\ker(h^{\mathbb{X}}) = \{(x_1, x_2) \in X^2 \mid h^{\mathbb{X}}(x_1) = h^{\mathbb{X}}(x_2)\}.$$

2. If $\mathbb{M} = \langle M; p, q \rangle$ is a partial unary algebra, then the ternary relation

$$r := \{\, (a, b, c) \in M^3 \mid p(a) = b \;\&\; q(c) = q(c) \,\} = \mathrm{graph}(p) \times \mathrm{dom}(q)$$

is conjunct-atomic definable from $\mathbb{M}$.

Definition 1.2.3. Let $\mathbf{M}$ be a finite algebra. For each $n \geqslant 1$, an n-ary relation r on M is **compatible with M** if it is a non-empty subuniverse of $\mathbf{M}^n$. Two compatible relations on $\mathbf{M}$ are **equivalent** if each is conjunct-atomic definable from the other. If the set of all compatible relations on $\mathbf{M}$ has a finite number of equivalence classes, then we say that **M admits only finitely many relations**. Otherwise, we say that **M admits infinitely many relations**.

Example 1.2.4. We first consider the 2-element bounded distributive lattice $\mathbf{2} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$, where $0 < 1$. The two relations

$$\leqslant := \{00, 01, 11\} \subseteq 2^2 \quad \text{and} \quad r := \{0000, 0100, 0011, 0111, 1111\} \subseteq 2^4,$$

given in Figure 1.1, are compatible with $\mathbf{2}$. The relations $\leqslant$ and r are equivalent, since they are interdefinable as follows:

$$\begin{aligned} \leqslant &= \{\, (x,y) \in 2^2 \mid (x,x,y,y) \in r \,\}; \\ r &= \{\, (x,y,z,w) \in 2^4 \mid x \leqslant y \;\&\; x \leqslant z \;\&\; z = w \,\}. \end{aligned}$$

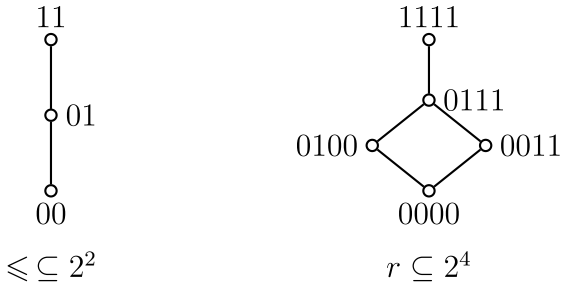


Figure 1.1: Two equivalent compatible relations on $\mathbf{2}$

In fact, every compatible relation on $\mathbf{2}$ is interdefinable in this way with either $\leqslant$ or the diagonal relation $\Delta_2 \subseteq 2^2$. Therefore the 2-element bounded lattice $\mathbf{2}$ has only two compatible relations, up to equivalence, and hence admits only finitely many relations (see [25, Example 2.9]).

For $n \geqslant 2$, an $(n+1)$-ary **near-unanimity** (NU) **term** t for $\mathbf{M}$ is an $(n+1)$-ary term such that $\mathbf{M}$ satisfies the identities

$$t(x,\dots,x,y) \approx t(x,\dots,x,y,x) \approx \quad \dots \quad \approx t(y,x,\dots,x) \approx x.$$

A ternary near-unanimity term for $\mathbf{M}$ is called a **majority term**. The following result from [24], a reinterpretation of part of the basic result from Baker and

Pixley [2, 2.1], shows that the existence of an $(n+1)$-ary NU term for a finite algebra **M** ensures that all compatible relations on **M** can be defined from a finite set of compatible relations.

Theorem 1.2.5 ([24, 1.3])**.** *Let* **M** *be a finite algebra and let* $n \in \mathbb{N}$*, with* $n \geqslant 2$*. Then the following are equivalent:*

(1) *every compatible relation on* **M** *is conjunct-atomic definable from the* n*-ary compatible relations on* **M***,*

(2) **M** *has an* $(n+1)$*-ary NU term.*

It follows from Theorem 1.2.5 that a finite algebra **M** that admits only finitely many relations must have a near-unanimity term.

Remark 1.2.6. Let **A** be a finite lattice-based algebra. Then **A** has a majority term which is the median term

$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Hence every compatible relation on **A** is conjunct-atomic definable from binary compatible relations on **A**, by Theorem 1.2.5.

A relation r on M is **directly decomposable** if, up to permutation of coordinates, it can be written as $p \times q$, for non-trivial relations p, q on M. Otherwise, the relation r is **directly indecomposable**. The following lemma is implicit in the proof of [25, Example 2.10].

Lemma 1.2.7. *Let* **M** *be a finite algebra. If the set of all directly indecomposable compatible relations on* **M** *has only a finite number of equivalence classes, then* **M** *admits only finitely many relations.*

Next we shall use a certain structure related to a finite algebra to represent its compatible relations.

Definition 1.2.8. Let **M** be a finite algebra and let $\mathbb{M} = \langle M; G, H, R \rangle$ be a structure on the same underlying set. The structure $\mathbb{M}$ is called an **alter ego**

of $\mathbf{M}$ if each relation in the set $R \cup \{\, \mathrm{graph}(f) \mid f \in G \cup H \,\}$ is compatible with $\mathbf{M}$.

Let $\mathbb{M}$ be an alter ego of a finite algebra $\mathbf{M}$. We use $\mathsf{ISP_f}(\mathbb{M})$ to denote the class of all isomorphic copies of non-empty substructures of non-zero finite powers of $\mathbb{M}$. For each structure $\mathbb{X}$ in $\mathsf{ISP_f}(\mathbb{M})$ and each non-empty subset S of X, we define the relation

$$\mathrm{E}(\mathbb{X}){\restriction}_S := \left\{\, \alpha{\restriction}_S \;\middle|\; \alpha\colon \mathbb{X} \to \mathbb{M} \,\right\} \subseteq M^S.$$

Note that the relation $\mathrm{E}(\mathbb{X}){\restriction}_S$ is compatible with $\mathbf{M}$ because $\mathbb{M}$ is an alter ego. In some cases, we can use a structure $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ in tandem with a non-empty set $S \subseteq X$ to encode a given compatible relation on $\mathbf{M}$.

Example 1.2.9. We continue Example 1.2.4. The ordered set $\mathbb{2} = \langle \{0,1\}; \leqslant \rangle$, where $0 \leqslant 1$, is an alter ego for the 2-element bounded distributive lattice $\mathbf{2} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$. Consider the ordered set $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{2})$ shown in Figure 1.2 below. A morphism from $\mathbb{X}$ to $\mathbb{2}$ is an order-preserving map. Let $S = X$. Then we can work out the relation

$$\begin{aligned} \mathrm{E}(\mathbb{X}){\restriction}_S &= \left\{\, \alpha{\restriction}_S \;\middle|\; \alpha\colon \mathbb{X} \to \mathbb{2} \,\right\} \\ &= \left\{ 000, 010, 001, 011, 111 \,\right\} \subseteq 2^3. \end{aligned}$$

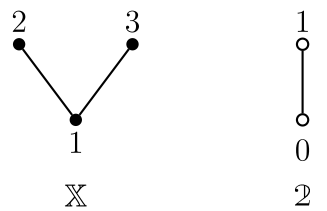


Figure 1.2:

This relation is equivalent to the relation r from Figure 1.1 (by removing the repeated coordinate). Hence the compatible relation r is encoded by the structure $\mathbb{X}$ and the set S.

It is natural to ask the question: "Under what conditions is every compatible relation on a finite algebra $\mathbf{M}$ encoded in the form $\mathrm{E}(\mathbb{X}){\restriction_S}$?". In fact, we wish to have encodings with the stronger property that S is a generating set for $\mathbb{X}$. By the following lemma, an alter ego $\mathbb{M}$ can be used to encode, in this stronger way, all compatible relations on the algebra $\mathbf{M}$ provided the following interpolation condition holds:

(IC) for all $n \geqslant 1$ and all $\mathbb{X} \leqslant \mathbb{M}^n$, every morphism $\alpha\colon \mathbb{X} \to \mathbb{M}$ extends to an n-ary term function of the algebra $\mathbf{M}$.

Lemma 1.2.10 ([24, 2.3]). *Let $\mathbf{M}$ be a finite algebra and let $\mathbb{M}$ be an alter ego of $\mathbf{M}$ such that* (IC) *holds. Then each compatible relation on $\mathbf{M}$ is equivalent to one of the form*

$$\mathrm{E}(\mathbb{X}){\restriction_S} := \{\, \alpha{\restriction_S} \mid \alpha\colon \mathbb{X} \to \mathbb{M} \,\} \subseteq M^S,$$

where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a non-empty generating set for $\mathbb{X}$.

In this thesis, we shall also use the following equivalent version of (IC).

Lemma 1.2.11 ([25, 4.5]). *Let $\mathbf{M}$ be a finite algebra and let $\mathbb{M}$ be an alter ego of $\mathbf{M}$. Then* (IC) *holds if and only if every compatible relation on $\mathbf{M}$ is conjunct-atomic definable from $\mathbb{M}$.*

The following two results will be used to show that one compatible relation is conjunct-atomic definable from another.

Lemma 1.2.12 ([24, 2.6]). *Let $\mathbf{M}$ be a finite algebra. Let $\mathrm{E}(\mathbb{X}){\restriction_S}$ and $\mathrm{E}(\mathbb{Y}){\restriction_T}$ be compatible relations on $\mathbf{M}$, associated with an alter ego $\mathbb{M}$, such that T is a generating set for $\mathbb{Y}$. Then $\mathrm{E}(\mathbb{X}){\restriction_S}$ is conjunct-atomic definable from $\mathrm{E}(\mathbb{Y}){\restriction_T}$ if and only if the following holds:*

- *for each map $\varphi\colon S \to M$ that does not extend to a morphism from $\mathbb{X}$ to $\mathbb{M}$, there exists a morphism $\omega\colon \mathbb{Y} \to \mathbb{X}$ with $\omega(T) \subseteq S$ such that the map $\varphi \circ \omega{\restriction_T}\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$.*

Lemma 1.2.13 ([24, 5.1])**.** *Let* $\mathbf{M}$ *be a finite algebra. Let* $\mathrm{E}(\mathbb{X}){\restriction}_S$ *and* $\mathrm{E}(\mathbb{Y}){\restriction}_T$ *be compatible relations on* $\mathbf{M}$*, associated with an alter ego* $\mathbb{M}$*, such that* S *is a generating set for* $\mathbb{X}$*. Assume* $\mathbb{Y} \leqslant \mathbb{X}$ *and there is a retraction* $\rho\colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ *with* $\rho{\restriction}_Y = \mathrm{id}_Y$ *and* $\rho(S) \subseteq T \subseteq S$*. Then* $\mathrm{E}(\mathbb{Y}){\restriction}_T$ *is conjunct-atomic definable from* $\mathrm{E}(\mathbb{X}){\restriction}_S$*.*

Let $\mathbb{X}$ and $\mathbb{Y}$ be two structures of the same type. We say that $\mathbb{X}$ is a **divisor** of $\mathbb{Y}$ if $\mathbb{X} \in \mathsf{HS}(\mathbb{Y})$. We shall employ the following useful result from [24] to show that a finite algebra admits infinitely many compatible relations.

Transfer Lemma 1.2.14 ([24, 3.3])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite algebras such that* $\mathbf{A}$ *is a divisor of* $\mathbf{B}$*. If* $\mathbf{A}$ *admits infinitely many relations, then so does* $\mathbf{B}$*.*

Example 1.2.15. By encoding compatible relations via structures and using these results, Davey and Pitkethly [24] obtain the following examples:

- a finite Boolean algebra $\mathbf{B}$ admits only finitely many relations if and only if $|B| \leqslant 2$;
- a finite lattice $\mathbf{L}$ admits only finitely many relations if and only if $|L| \leqslant 2$;
- a finite Heyting algebra that is not a chain admits infinitely many relations.

1.3 Natural dualities

Given a finite algebra $\mathbf{M}$, one of the most important techniques used to count the compatible relations on $\mathbf{M}$ is to find a suitable alter ego structure $\mathbb{M}$ for $\mathbf{M}$. The theory of natural dualities provides us practical ways to make this problem easier. In this section, we shall give a brief review of the aspects of natural dualities required in this thesis. We refer the reader to the text by Clark and Davey [13] for more details and results in the theory of natural dualities.

Let $\mathbf{M}$ be a finite algebra and let $\boldsymbol{\mathcal{A}} := \mathsf{ISP}(\mathbf{M})$. Then $\boldsymbol{\mathcal{A}}$ is the quasi-variety generated by $\mathbf{M}$; see Theorem [13, 1.3.4]. Let $\mathbb{M} = \langle M; G, H, R\rangle$ be an alter ego of $\mathbf{M}$ and let $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ be the topological structure obtained by adding the discrete topology $\mathcal{T}$ on M to the structure $\mathbb{M}$. Define $\boldsymbol{\mathcal{X}} := \mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{M}_\mathcal{T})$ to be the class of all isomorphic copies of (possibly empty) topologically closed substructures of non-zero powers of $\mathbb{M}_\mathcal{T}$. The structure on the powers of $\mathbb{M}_\mathcal{T}$ is lifted pointwise from the structure of $\mathbb{M}_\mathcal{T}$. The topology on the powers of $\mathbb{M}_\mathcal{T}$ is the usual product topology lifted from the discrete topology on $\mathbb{M}_\mathcal{T}$. The morphisms in $\boldsymbol{\mathcal{X}}$ are continuous structure-preserving maps.

Because the structures in the type of $\mathbb{M}_\mathcal{T}$ are compatible with the algebra $\mathbf{M}$, there is a natural dual adjunction between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$. The contravariant functors $D\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}$ and $E\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$ are given as follows.

- For each algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, **the dual of** $\mathbf{A}$ is the closed substructure $D(\mathbf{A})$ of $(\mathbb{M}_\mathcal{T})^A$ whose the underlying set is the set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}$.
- For each structure $\mathbb{X} \in \boldsymbol{\mathcal{X}}$, **the dual of** $\mathbb{X}$ is the subalgebra $E(\mathbb{X})$ of $\mathbf{M}^X$ whose the underlying set is the set $\boldsymbol{\mathcal{X}}(\mathbb{X}, \mathbb{M}_\mathcal{T})$ of all morphisms from $\mathbb{X}$ to $\mathbb{M}_\mathcal{T}$.
- For each homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ in $\boldsymbol{\mathcal{A}}$, the morphism $D(\varphi)\colon D(\mathbf{B}) \to D(\mathbf{A})$ is given by $D(\varphi)(x) := x \circ \varphi$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \mathbf{M})$.
- For each morphism $\psi\colon \mathbb{X} \to \mathbb{Y}$ in $\boldsymbol{\mathcal{X}}$, the homomorphism $E(\psi)\colon E(\mathbb{Y}) \to E(\mathbb{X})$ is given by $E(\psi)(\alpha) := \alpha \circ \psi$, for all $\alpha \in \boldsymbol{\mathcal{X}}(\mathbb{Y}, \mathbb{M}_\mathcal{T})$.

Now for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, define the **evaluation map** $e_\mathbf{A}\colon \mathbf{A} \to ED(\mathbf{A})$ by $e_\mathbf{A}(a)(x) := x(a)$, for all $a \in A$ and all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. Similarly, for each $\mathbb{X} \in \boldsymbol{\mathcal{X}}$, define the **evaluation map** $\varepsilon_\mathbb{X}\colon \mathbb{X} \to DE(\mathbb{X})$ by $\varepsilon_\mathbb{X}(x)(\alpha) := \alpha(x)$, for all $x \in X$ and all $\alpha \in \boldsymbol{\mathcal{X}}(\mathbb{X}, \mathbb{M}_\mathcal{T})$. The maps $e_\mathbf{A}$ and $\varepsilon_\mathbb{X}$ are automatically embeddings.

Definition 1.3.1. If the evaluation map $e_\mathbf{A}$ is an isomorphism for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, then we say that $\mathbb{M}_\mathcal{T}$ **yields a (natural) duality on** $\boldsymbol{\mathcal{A}}$ or, alternatively, that $G \cup H \cup R$ **yields a duality on** $\boldsymbol{\mathcal{A}}$. If both $e_\mathbf{A}$ and $\varepsilon_\mathbb{X}$ are isomorphisms, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $\mathbb{X} \in \boldsymbol{\mathcal{X}}$, then we say that $\mathbb{M}_\mathcal{T}$ **yields a full duality on** $\boldsymbol{\mathcal{A}}$. In this case, the categories $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}$ are dually equivalent. The finite algebra $\mathbf{M}$

is called **(fully) dualisable** if there is an alter ego $\mathbb{M}_{\mathcal{T}}$ of $\mathbf{M}$ which yields a (full) duality on $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$. If the endomorphisms of $\mathbf{M}$ yield a duality on $\boldsymbol{\mathcal{A}}$, then we say that $\mathbf{M}$ is **endodualisable**.

Pontryagin duality for abelian groups of exponent m [55], Stone duality for Boolean algebras [60] and Priestley duality for bounded distributive lattices [56, 57] are classical examples of natural dualities. In this thesis, all finite algebras investigated are bounded distributive lattice-based algebras and hence Priestley duality will be used. Priestley duality sets up a dual equivalence between the category $\mathfrak{D}$ of bounded distributive lattices and a special category $\mathfrak{P}$ of topological ordered spaces. As Priestley duality plays an important role in this thesis, we shall present it here in more details. We first introduce the following notions.

By a **topological ordered set** $\mathbb{X} := \langle X; \leqslant, \mathcal{T}\rangle$, we mean a set X endowed with a order relation $\leqslant$ and a topology $\mathcal{T}$. A subset U of X is called a down-set in X if, for all $x \in U$ and $y \in X$, if $y \leqslant x$ then $y \in U$. An up-set in X is dually defined. We say that the topological ordered set $\mathbb{X} := \langle X; \leqslant, \mathcal{T}\rangle$ is **totally order-disconnected** if for all $x, y \in X$ with $x \nleqslant y$ there exists a clopen down-set U such that $x \notin U$ and $y \in U$. A compact totally order-disconnected space is called a **Priestley space**. The following are some facts about Priestley spaces used in this thesis.

Remark 1.3.2. Let $\mathbb{X} = \langle X; \leqslant, \mathcal{T}\rangle$ be a Priestley space.

(1) Every element of $\mathbb{X}$ is greater than or equal to a minimal element of $\mathbb{X}$ (see [28, Exercise 11.15(i)]).

(2) For all $x \in X$, the up-set ${\uparrow}x := \{y \in X \mid y \geqslant x\}$ and the down-set ${\downarrow}x := \{y \in X \mid y \leqslant x\}$ are closed (see [28, Exercise 11.14(i)]).

(3) If Y is a closed down-set of $\mathbb{X}$ and Z is a closed up-set of $\mathbb{X}$ with $Y \cap Z = \varnothing$, then there is a clopen up-set U of $\mathbb{X}$ with $Z \subseteq U$ and $Y \cap U = \varnothing$ (see [28, Lemma 11.21(ii)]).

Example 1.3.3 (Priestley duality for bounded distributive lattices). Let $\mathbf{2} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ be the 2-element bounded distributive lattice and define $\boldsymbol{\mathcal{D}} := \mathsf{ISP}(\mathbf{2})$. Then $\boldsymbol{\mathcal{D}}$ is the category of all bounded distributive lattices. The topological ordered set $\mathbb{2}_{\mathcal{T}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$ is an alter ego of $\mathbf{2}$, where $0 \leqslant 1$ and $\mathcal{T}$ is the discrete topology. Define $\boldsymbol{\mathcal{P}} := \mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{2})$. Then $\boldsymbol{\mathcal{P}}$ is the category of all Priestley spaces. The morphisms in $\boldsymbol{\mathcal{P}}$ are continuous order-preserving maps. Priestley [56] showed that $\mathbb{2}_{\mathcal{T}}$ yields a full duality on $\boldsymbol{\mathcal{D}}$. The associated contravariant functors are usually denoted by $H \colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{P}}$ and $K \colon \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{D}}$.

Definition 1.3.4. Let $\mathbf{M}$ be a finite algebra. Let $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}$ and let $\boldsymbol{\mathcal{X}} = \mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{M}_{\mathcal{T}})$. The structure $\mathbb{M}_{\mathcal{T}}$ is **injective** in $\boldsymbol{\mathcal{X}}$ if, for every morphism $\alpha \colon \mathbb{X} \to \mathbb{M}_{\mathcal{T}}$ and embedding $\varphi \colon \mathbb{X} \to \mathbb{Y}$ in $\boldsymbol{\mathcal{X}}$, there is a morphism $\beta \colon \mathbb{Y} \to \mathbb{M}_{\mathcal{T}}$ such that $\beta \circ \varphi = \alpha$.

The injectivity of $\mathbf{M}$ in $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$ is defined similarly, that is, the algebra $\mathbf{M}$ is **injective** in $\boldsymbol{\mathcal{A}}$ if, for every homomorphism $\alpha \colon \mathbf{A} \to \mathbf{M}$ and embedding $\varphi \colon \mathbf{A} \to \mathbf{B}$ in $\boldsymbol{\mathcal{A}}$, there is a homomorphism $\beta \colon \mathbf{B} \to \mathbf{M}$ such that $\beta \circ \varphi = \alpha$. Observe that, if $\mathbf{M}$ is injective in $\boldsymbol{\mathcal{A}}$, then every partial endomorphism of $\mathbf{M}$ extends to a total endomorphism of $\mathbf{M}$.

Definition 1.3.5. Let $\mathbf{M}$ be a finite algebra and $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ be an alter ego of $\mathbf{M}$. We say that $\mathbb{M}_{\mathcal{T}}$ **yields a strong duality on** $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$ if $\mathbb{M}_{\mathcal{T}}$ yields a full duality on $\boldsymbol{\mathcal{A}}$ and $\mathbb{M}_{\mathcal{T}}$ is injective in $\boldsymbol{\mathcal{X}} = \mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{M}_{\mathcal{T}})$.

Note that, if $\mathbb{M}_{\mathcal{T}}$ yields a strong duality on $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$, then $\mathbb{M}_{\mathcal{T}}$ satisfies the interpolation condition (IC) ([13, Lemma 2.2.5]). In some cases, it is more practical to obtain an alter ego $\mathbb{M}_{\mathcal{T}}$ of $\mathbf{M}$ which yields a strong duality on $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$ than to check the condition (IC) directly. The next result shows us how to do this for a certain class of algebras.

A non-trivial algebra $\mathbf{M}$ is said to be **subdirectly irreducible** if it has a smallest non-trivial congruence. Finite subdirectly irreducible algebras play a very important role in this thesis. The following result shows us how to upgrade a duality to a strong duality, in the case that every subalgebra of $\mathbf{M}$ is subdirectly irreducible. This result is a special case of [13, Theorem 3.3.7].

Theorem 1.3.6. *Let* $\mathbf{M}$ *be a finite algebra with a ternary near-unanimity term such that every non-trivial subalgebra of* $\mathbf{M}$ *is subdirectly irreducible. Assume that* $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ *yields a duality on* $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$*. If* $\mathbb{M}'_{\mathcal{T}}$ *is obtained from* $\mathbb{M}_{\mathcal{T}}$ *by adding to* $G \cup H$ *all partial endomorphisms of* $\mathbf{M}$*, then* $\mathbb{M}'_{\mathcal{T}}$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$*.*

The following result, a part of the result from [13, Lemma 3.2.3], shows that we only need the maximal partial endomorphisms.

Lemma 1.3.7. *Let* $\mathbf{M}$ *be a finite algebra and assume* $\mathbb{M}'_{\mathcal{T}} = \langle M; G', H', R', \mathcal{T}\rangle$ *yields a strong duality on* $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$*. Let* $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ *be a structure obtained from* $\mathbb{M}'_{\mathcal{T}}$ *by deleting members of* H' *which have an extension in* $G \cup H$*. If* $\mathbb{M}_{\mathcal{T}}$ *yields a duality on* $\boldsymbol{\mathcal{A}}$*, then* $\mathbb{M}_{\mathcal{T}}$ *also yields a strong duality on* $\boldsymbol{\mathcal{A}}$*.*

The previous two results lead to the following corollary.

Corollary 1.3.8. *Let* $\mathbf{M}$ *be a finite algebra with a ternary near-unanimity term such that every non-trivial subalgebra of* $\mathbf{M}$ *is subdirectly irreducible. Assume that* $\mathbb{M}_{\mathcal{T}} = \langle M; G, H, R, \mathcal{T}\rangle$ *yields a duality on* $\boldsymbol{\mathcal{A}} = \mathsf{ISP}(\mathbf{M})$ *and that* $\mathbf{M}$ *is injective in* $\boldsymbol{\mathcal{A}}$*. Then* $\mathbb{M}'_{\mathcal{T}} = \langle M; G \cup \mathrm{End}(\mathbf{M}), H, R, \mathcal{T}\rangle$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$*, where* $\mathrm{End}(\mathbf{M})$ *is the set of endomorphisms of* $\mathbf{M}$*.*

For a finite algebra $\mathbf{M}$, whenever the alter ego $\mathbb{M}_{\mathcal{T}}$ used to encode compatible relations on $\mathbf{M}$ satisfies (IC), we have, by Lemma 1.2.10, that every compatible relation on $\mathbf{M}$ is equivalent to one of the form $\mathrm{E}(\mathbb{X}){\restriction_S}$, where $\mathbb{X} \in \mathsf{ISP}_{\mathsf{f}}(\mathbb{M})$ and S is a non-empty generating set for $\mathbb{X}$. To understand the compatible relations of this form, we need to understand the finite structures in $\boldsymbol{\mathcal{X}} = \mathsf{IS}^0_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_{\mathcal{T}})$. The following theorem provides us a way to check when a structure is in the category $\boldsymbol{\mathcal{X}}$.

Theorem 1.3.9. (**Separation Theorem** [13, 1.4.4]) *Let* $\mathbb{M}_{\mathcal{T}}$ *be a finite topological structure of type* $\langle G, H, R\rangle$ *and let* $\mathbb{X}$ *be a non-trivial compact topological structure of the same type. Then* $\mathbb{X} \in \mathsf{IS}^0_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_{\mathcal{T}})$ *if and only if the following conditions hold:*

(1) *for each* $x, y \in X$ *with* $x \neq y$*, there is a morphism* $\alpha\colon \mathbb{X} \to \mathbb{M}_{\mathcal{T}}$ *such that* $\alpha(x) \neq \alpha(y)$*,*

(2) *for each* n*-ary* $h \in H$ *and* $(x_1, x_2, \ldots, x_n) \notin \mathrm{dom}(h^{\mathbb{X}})$*, there is a morphism* $\alpha\colon \mathbb{X} \to \mathbb{M}_{\mathcal{T}}$ *such that* $(\alpha(x_1), \ldots, \alpha(x_n)) \notin \mathrm{dom}(h^{\mathbb{M}_{\mathcal{T}}})$*, and*

(3) *for each* n*-ary* $r \in R$ *and* $(x_1, x_2, \ldots, x_n) \notin r^{\mathbb{X}}$*, there is a morphism* $\alpha\colon \mathbb{X} \to \mathbb{M}_{\mathcal{T}}$ *such that* $(\alpha(x_1), \ldots, \alpha(x_n)) \notin r^{\mathbb{M}_{\mathcal{T}}}$*.*

If (3) *holds for a fixed relation* r*, then we say that the morphisms from* $\mathbb{X}$ *to* $\mathbb{M}_{\mathcal{T}}$ ***separate*** r*.*

Note that we can omit the topology at the finite level, since all finite structures in $\mathsf{IS}_{\mathsf{c}}^{0}\mathsf{P}^{+}(\mathbb{M}_{\mathcal{T}})$ carry the discrete topology.

Chapter 2

Compatible relations on finite Heyting algebras

Intuitionistic logic is derived from the classical two-valued (Boolean) logic by reinterpreting the meaning of "or". Intuitionistic logic is modelled by Heyting algebras in the same way that classical logic is modelled by Boolean algebras. The study of the variety of Heyting algebras has been widely carried out by numerous authors. The basic theory of Heyting algebras may be found in the text by Balbes and Dwinger [3] and is also presented in the text by Rasiowa and Sikorski [58], in which they are referred to as pseudo-Boolean algebras. Heyting algebras, especially Heyting chains, have played an important role in the development of natural duality; see [15], [31], [32], [33], [27], [29], and [30].

A **Heyting algebra** is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ in which $\mathbf{A}^\flat = \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with a relative pseudo-complementation operation $\rightarrow$, that is,

$$a \rightarrow b = \max\{\, x \in A \mid x \wedge a \leqslant b \,\},$$

for all $a, b \in A$. The class of all Heyting algebras is an arithmetical variety; see [12]. (A variety is **arithmetical** if it is both congruence distributive and congruence permutable.)

In [24], Davey and Pitkethly showed that a finite Heyting algebra that is not a chain admits infinitely many relations and they conjectured that a finite Heyting chain admits only finitely many relations. In this chapter, we shall prove this conjecture, by giving a general sufficient condition for a finite algebra to admit only finitely many relations (Theorem 2.3.5) and showing that finite Heyting chains satisfy this condition. On the way, we introduce the algebraic property **strictly endoprimal** (Section 2.1). This property is related to the much-studied property 'endoprimal' [42, 44, 23], in the same way that 'strictly affine complete' is related to 'affine complete'. We also generalise the property **linear** from unary algebras to partial unary algebras (Section 2.2). Linear unary algebras have arisen previously in the study of group universality [59], decidable first-order theories [61] and natural dualities [53, 54].

2.1 Strict endoprimality

An algebra $\mathbf{M}$ is **endoprimal** if, for all $n \geqslant 1$, each operation $f\colon M^n \to M$ that preserves the endomorphisms of $\mathbf{M}$ is a term function of $\mathbf{M}$. There has been considerable interest in characterising the endoprimal algebras within various classes of algebras: for example, distributive lattices [48], Stone algebras [18, 23, 44], double Stone algebras [37], implication algebras [52] and abelian groups [40, 41, 43, 35]. See the survey paper [42] for a history of this topic.

Definition 2.1.1. Let $\mathbf{M}$ be a finite algebra. Recall that a partial endomorphism of $\mathbf{M}$ is a homomorphism $p\colon \mathbf{A} \to \mathbf{M}$, for some $\mathbf{A} \leqslant \mathbf{M}$. We shall use $\mathrm{End}(\mathbf{M})$ and $\mathrm{End}_{\mathrm{p}}(\mathbf{M})$ to denote the sets of all endomorphisms and all partial endomorphisms of $\mathbf{M}$, respectively. We say that $\mathbf{M}$ is **strictly endoprimal** if the alter ego $\mathbb{M} := \langle M; \mathrm{End}(\mathbf{M})\rangle$ satisfies (IC), that is,

- for all $n \geqslant 1$ and all $\mathbb{X} \leqslant \mathbb{M}^n$, every morphism $\alpha\colon \mathbb{X} \to \mathbb{M}$ extends to an n-ary term function of the algebra $\mathbf{M}$.

We say that $\mathbf{M}$ is **strictly p-endoprimal** if $\mathbb{M} := \langle M; \mathrm{End}_{\mathrm{p}}(\mathbf{M})\rangle$ satisfies (IC). So an algebra that is strictly endoprimal is also strictly p-endoprimal.

Example 2.1.2. Consider the Heyting chain $\mathbf{C}_3 = \langle \{0, c, 1\}; \vee, \wedge, \to, 0, 1\rangle$ with $0 < c < 1$. The only non-identity endomorphism e of $\mathbf{C}_3$ is given by $e(0) = 0$ and $e(c) = e(1) = 1$. Define the alter ego $\mathbb{C}_3 := \langle \{0, c, 1\}; e\rangle$ of $\mathbf{C}_3$; see Figure 2.1. We shall see in Lemma 2.1.13 that $\mathbb{C}_3$ satisfies (IC), and hence $\mathbf{C}_3$ is strictly endoprimal. (This example also arises in natural duality theory [13, 4.2.3].) By Lemma 1.2.10, each compatible relation on $\mathbf{C}_3$ can be encoded by a structure $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{C}_3)$ and generating set $\varnothing \neq S \subseteq X$. Note that $\mathsf{ISP_f}(\mathbb{C}_3)$ consists of all finite unars $\mathbb{X} = \langle X; e\rangle$ such that $e \circ e = e$. Figure 2.1 gives encodings of five compatible relations on $\mathbf{C}_3$; the generating sets are shaded. (In fact, each compatible relation on $\mathbf{C}_3$ is equivalent to one of these five relations. But we shall prove that $\mathbf{C}_3$ admits only finitely many relations via a more general result.)

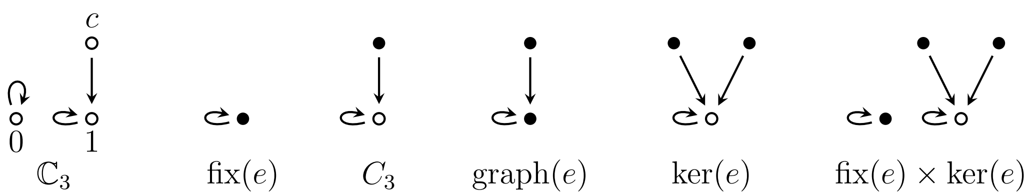


Figure 2.1: Five compatible relations on $\mathbf{C}_3$

The next lemma shows that strict p-endoprimality is a very restrictive property. Recall that a ternary term t is a **Pixley term** for an algebra if the equations $t(x, y, y) \approx t(x, y, x) \approx t(y, y, x) \approx x$ hold. An algebra generates an arithmetical variety if and only if it has a Pixley term; see [12].

Lemma 2.1.3 ([13, 6.4.1])**.** *Each strictly p-endoprimal algebra has a Pixley term.*

Proof. Let $\mathbf{M}$ be a finite algebra such that $\mathbb{M} := \langle M; \mathrm{End_p}(\mathbf{M})\rangle$ satisfies (IC). Define $X := \{\, (x, y, y), (x, y, x), (y, y, x) \mid x, y \in M \,\}$. Then $\mathbb{X} \leqslant \mathbb{M}^3$, as X is closed under all partial unary operations on M. Now let $\alpha\colon X \to M$ be the restricted Pixley operation, i.e.,

$$\alpha(x, y, y) = \alpha(x, y, x) = \alpha(y, y, x) = x,$$

for all $x, y \in M$. Then it is easy to verify that α preserves all partial unary operations on M, and hence $\alpha\colon \mathbb{X} \to \mathbb{M}$ is a morphism. It follows that α extends to a term function of $\mathbf{M}$, as $\mathbb{M}$ satisfies (IC). So $\mathbf{M}$ has a Pixley term, as required. □

We shall give examples of strictly endoprimal and strictly p-endoprimal algebras using the following lemma. Note that this lemma does not hold for 'endoprimal': the 2-element lattice $\mathbf{2}$ and the 3-element lattice $\mathbf{3}$ both generate the quasi-variety of distributive lattices, but $\mathbf{3}$ is endoprimal and $\mathbf{2}$ is not (Márki and Pöschel [48]).

Lemma 2.1.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite algebras such that* $\mathsf{ISP}(\mathbf{A}) = \mathsf{ISP}(\mathbf{B})$.

(1) *If* $\mathbf{A}$ *is strictly endoprimal, then so is* $\mathbf{B}$.

(2) *If* $\mathbf{A}$ *is strictly p-endoprimal, then so is* $\mathbf{B}$.

Proof. We can assume that $\mathbf{B} \leqslant \mathbf{A}^m$, for some $m \geqslant 1$. We shall prove (2); the proof of (1) is almost identical. Assume that $\mathbf{A}$ is strictly p-endoprimal. Then every compatible relation on $\mathbf{A}$ is conjunct-atomic definable from $\mathrm{End_p}(\mathbf{A})$, by Lemma 1.2.11.

Now let $\mathbf{r} \leqslant \mathbf{B}^n$, for some $n \geqslant 1$. We want to show that r is conjunct-atomic definable from $\mathrm{End_p}(\mathbf{B})$. We have $\mathbf{r} \leqslant (\mathbf{A}^m)^n$, and so r is conjunct-atomic definable from $\mathrm{End_p}(\mathbf{A})$:

$$r = \left\{\, (b_1, \dots, b_n) \in (A^m)^n \;\middle|\; (\forall s \in S)\ p_s(b_{i_s}(j_s)) = q_s(b_{k_s}(\ell_s)) \,\right\},$$

for some $p_s, q_s \in \mathrm{End_p}(\mathbf{A})$, $i_s, k_s \leqslant n$ and $j_s, \ell_s \leqslant m$ indexed over a finite set S.

Since $\mathbf{B} \leqslant \mathbf{A}^m$, we can define the restricted projection $\pi_i\colon \mathbf{B} \to \mathbf{A}$, for each $i \leqslant m$. Since $\mathbf{A} \in \mathsf{ISP}(\mathbf{B})$, the set Σ of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$

separates the elements of A. It now follows that

$$r = \big\{ (b_1, \dots, b_n) \in B^n \;\big|\; (\forall s \in S)(\forall \sigma \in \Sigma)\ \sigma \circ p_s \circ \pi_{j_s}(b_{i_s}) = \sigma \circ q_s \circ \pi_{\ell_s}(b_{k_s}) \big\},$$

where each composition has the maximum possible domain. Note that both $\sigma \circ p_s \circ \pi_{j_s}$ and $\sigma \circ q_s \circ \pi_{\ell_s}$ belong to $\mathrm{End_p}(\mathbf{B})$, for all $s \in S$ and $\sigma \in \Sigma$. So r is conjunct-atomic definable from $\mathrm{End_p}(\mathbf{B})$. Hence $\mathbf{B}$ is strictly p-endoprimal, by Lemma 1.2.11. □

Example 2.1.5. *Every finite Boolean algebra is strictly endoprimal.*

Proof. Let $\mathbf{B} = \langle \{0, 1\}; \vee, \wedge, ', 0, 1 \rangle$ be the 2-element Boolean algebra. Every finitary partial operation on $\{0, 1\}$ extends to a term function of $\mathbf{B}$, and therefore $\mathbf{B}$ is strictly endoprimal. So every finite Boolean algebra is strictly endoprimal, by Lemma 2.1.4. □

We shall use the following lemma to give examples of Heyting algebras that are strictly p-endoprimal. This lemma is based on the Heyting Strong Duality Theorem [13, 4.2.2], which has an extra assumption (that $\mathbf{M}$ is subdirectly irreducible) and a stronger conclusion. We give a direct proof that avoids the machinery of natural duality theory. To prove the lemma, we use the following useful result due to Fleischer.

Theorem 2.1.6 (Fleischer's Theorem [34])**.** *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be algebras such that congruences on subalgebras of $\mathbf{M}_1 \times \mathbf{M}_2$ permute. Then every subalgebra $\mathbf{A}$ of $\mathbf{M}_1 \times \mathbf{M}_2$ is a joint kernel $\ker(f_1, f_2)$ for some algebra $\mathbf{C}$, some subalgebras $\mathbf{B}_i \leqslant \mathbf{M}_i$ and some surjections $f_i\colon \mathbf{B}_i \twoheadrightarrow \mathbf{C}$, that is,*

$$A = \ker(f_1, f_2) := \{ (x, y) \in B_1 \times B_2 \mid f_1(x) = f_2(y) \}.$$

Lemma 2.1.7. *A finite Heyting algebra $\mathbf{M}$ is strictly p-endoprimal provided $\mathsf{HS}(\mathbf{M}) \subseteq \mathsf{ISP}(\mathbf{M})$.*

Proof. Assume $\mathbf{M}$ is a finite Heyting algebra such that $\mathsf{HS}(\mathbf{M}) \subseteq \mathsf{ISP}(\mathbf{M})$. Each compatible relation on $\mathbf{M}$ is conjunct-atomic definable from the set of all binary compatible relations on $\mathbf{M}$, by Remark 1.2.6, as $\mathbf{M}$ is lattice-based. To show that $\mathbf{M}$ is strictly p-endoprimal, using Lemma 1.2.11, it suffices to show that every binary compatible relation on $\mathbf{M}$ is conjunct-atomic definable from $\mathrm{End}_{\mathrm{p}}(\mathbf{M})$.

Let $\mathbf{r} \leqslant \mathbf{M}^2$. As Heyting algebras have permuting congruences, we can use Fleischer's Theorem 2.1.6. So there are surjections $f_i \colon \mathbf{B}_i \twoheadrightarrow \mathbf{C}$ with $\mathbf{B}_i \leqslant \mathbf{M}$, for $i \in \{1, 2\}$, such that

$$r = \ker(f_1, f_2) := \{\, (x, y) \in B_1 \times B_2 \mid f_1(x) = f_2(y) \,\}.$$

Since $\mathbf{C} \in \mathsf{HS}(\mathbf{M}) \subseteq \mathsf{ISP}(\mathbf{M})$, the set Σ of all homomorphisms from $\mathbf{C}$ to $\mathbf{M}$ separates the elements of C. So

$$r = \{\, (x, y) \in M^2 \mid (\forall \sigma \in \Sigma)\ \sigma \circ f_1(x) = \sigma \circ f_2(y) \,\}.$$

But $\sigma \circ f_i \colon \mathbf{B}_i \to \mathbf{M}$ is a partial endomorphism of $\mathbf{M}$, for all $\sigma \in \Sigma$ and $i \in \{1, 2\}$. Thus r is conjunct-atomic definable from $\mathrm{End}_{\mathrm{p}}(\mathbf{M})$, as required. □

For each $n \geqslant 1$, the n-element chain $\mathbf{C}_n = \langle \{c_1, c_2, \dots, c_n\}; \vee, \wedge, \to, 0, 1 \rangle$ is a Heyting algebra, where $0 = c_1 < c_2 < \cdots < c_n = 1$. A Heyting algebra that satisfies the equation $(x \to y) \vee (y \to x) \approx 1$ is called a **relative Stone algebra**. Hecht and Katriňák [38] showed that a finite Heyting algebra is a relative Stone algebra if and only if it belongs to $\mathsf{ISP}(\mathbf{C}_n)$, for some $n \geqslant 1$. This leads to the following result.

Lemma 2.1.8. *Let* $\mathbf{A}$ *be a finite relative Stone algebra. Then* $\mathbf{A}$ *generates the same quasi-variety as* $\mathbf{C}_n$*, for some* $n \geqslant 1$*, i.e.,* $\mathsf{ISP}(\mathbf{A}) = \mathsf{ISP}(\mathbf{C}_n)$*.*

Proof. Assume that $\mathbf{A}$ is a finite relative Stone algebra. Choose the smallest $n \geqslant 1$ such that $\mathbf{A} \in \mathsf{ISP}(\mathbf{C}_n)$. Then there must be a surjection $\varphi \colon \mathbf{A} \twoheadrightarrow \mathbf{C}_n$. It then follows that there is an embedding $\psi \colon \mathbf{C}_n \rightarrowtail \mathbf{A}$, as $\mathbf{C}_n$ is a projective Heyting algebra [4]. (Alternatively, an embedding is given by $\psi(0) = 0$ and $\psi(c) = \bigvee \varphi^{-1}(c)$, for all $c \in C_n \setminus \{0\}$.) So $\mathbf{C}_n \in \mathsf{ISP}(\mathbf{A})$, and hence we have

$\mathsf{ISP}(\mathbf{A}) = \mathsf{ISP}(\mathbf{C}_n)$, as required. □

Lemma 2.1.9. *Every finite relative Stone algebra is strictly p-endoprimal. In particular, every finite Heyting chain is strictly p-endoprimal.*

Proof. For each $n \geqslant 1$, the chain $\mathbf{C}_n$ is strictly p-endoprimal by Lemma 2.1.7, since $\mathsf{HS}(\mathbf{C}_n) \subseteq \mathsf{ISP}(\mathbf{C}_n)$. Now assume that $\mathbf{A}$ is a finite relative Stone algebra. Then $\mathsf{ISP}(\mathbf{A}) = \mathsf{ISP}(\mathbf{C}_n)$, for some $n \geqslant 1$, by Lemma 2.1.8. Since $\mathbf{C}_n$ is strictly p-endoprimal, it now follows by Lemma 2.1.4 that so is $\mathbf{A}$. □

Remark 2.1.10. A nice method for constructing examples of subdirectly irreducible Heyting algebras $\mathbf{M}$ that satisfy the condition $\mathsf{HS}(\mathbf{M}) \subseteq \mathsf{ISP}(\mathbf{M})$ of Lemma 2.1.7 was given by Davey and Werner [32]; see also [13, 7.3.2].

The finite Heyting chains were the first examples of endoprimal algebras (Davey [15]). Later, all the finite relative Stone algebras were shown to be endoprimal (Davey and Pitkethly [23]). We next describe which finite relative Stone algebras are strictly endoprimal.

Let $n \geqslant 1$. The relative pseudo-complementation on the Heyting chain $\mathbf{C}_n$ is given by

$$a \to b = \begin{cases} 1, & \text{if } a \leqslant b, \\ b, & \text{if } a > b, \end{cases}$$

for all $a, b \in C_n$. The next lemma describes the partial endomorphisms of $\mathbf{C}_n$.

Lemma 2.1.11. *Let $p\colon A \to C_n$ with $\{0,1\} \subseteq A \subseteq C_n$. Then p is a partial endomorphism of $\mathbf{C}_n$ if and only if*

(1) *$p(0) = 0$ and $p(1) = 1$,*

(2) *p is order-preserving, and*

(3) *p is one-to-one away from 1. That is, for all $a, b \in A$ with $a \neq b$, if $p(a) = p(b)$ then $p(a) = 1$.*

Proof. Assume that $p\colon \mathbf{A} \to \mathbf{C}_n$ is a partial endomorphism of $\mathbf{C}_n$. Then clearly (1) and (2) hold. To prove (3), let $a, b \in A$ with $a \neq b$ and $p(a) = p(b)$.

Since A is a chain, we may assume that $a > b$. Then $a \to b = b$ and hence $p(b) = p(a \to b) = p(a) \to p(b) = 1$, as $p(a) = p(b)$, by the assumption. Thus p is one-to-one away from 1.

Conversely, assume that $p\colon A \to C_n$ is a map satisfying all three conditions above. Then p preserves $\vee$ and $\wedge$, as $\mathbf{C}_n$ is a chain. To prove p is a partial endomorphism of $\mathbf{C}_n$, it suffices to prove that p preserves $\to$. Let $a, b \in A$. We consider the following two cases.

Case 1: $a \leqslant b$.

Then $p(a) \leqslant p(b)$ and $a \to b = 1$. It follows that $p(a) \to p(b) = 1$. We also have $p(a \to b) = p(1) = 1$. Hence $p(a) \to p(b) = p(a \to b)$.

Case 2: $a > b$.

Then $p(a) \geqslant p(b)$ and $a \to b = b$.

(i) If $p(a) = p(b)$, then $p(a) = p(b) = 1$ because p is one-to-one away from 1. It follows that $p(a) \to p(b) = 1$ and $p(a \to b) = p(b) = 1$. Therefore we have $p(a) \to p(b) = p(a \to b)$.

(ii) If $p(a) > p(b)$, then we have $p(a) \to p(b) = p(b) = p(a \to b)$.

Thus p preserves $\to$. Hence p is a partial endomorphism of $\mathbf{C}_n$. □

It follows from Lemma 2.1.11 that each endomorphism e of $\mathbf{C}_n$ is **increasing** (that is, $a \leqslant e(a)$, for all $a \in C_n$).

Lemma 2.1.12. *For $n \geqslant 5$, the n-element Heyting chain is not strictly endoprimal.*

Proof. Let $n \geqslant 5$ and let $\mathbf{C}_n = \langle \{c_1, c_2, \ldots, c_n\}; \vee, \wedge, \to, 0, 1\rangle$ denote the n-element Heyting chain, where $0 = c_1 < c_2 < \cdots < c_n = 1$. We now prove that the alter ego $\mathbb{C}_n := \langle C_n; \mathrm{End}(\mathbf{C}_n)\rangle$ of $\mathbf{C}_n$ does not satisfy (IC). We shall find $\mathbb{X} \leqslant \mathbb{C}_n^2$ and a morphism $\alpha\colon \mathbb{X} \to \mathbb{C}_n$ such that α does not extend to a binary term function of the Heyting chain $\mathbf{C}_n$.

Define $y := (c_2, c_{n-1}) \in C_n^2$ and define the principal filter $X := \uparrow y$ of $\mathbf{C}_n^2$. For all $e \in \mathrm{End}(\mathbf{C}_n)$ and $x \in X$, we have $e(x) \geqslant x \geqslant y$, since e is increasing. Therefore $\mathbb{X}$ is a substructure of $\mathbb{C}_n^2$.

Now define the map $\alpha\colon X \to C_n$ by

$$\alpha(x) := \begin{cases} 1, & \text{if } x \neq y, \\ c_{n-1}, & \text{if } x = y. \end{cases}$$

To see that α is a morphism, let $e \in \mathrm{End}(\mathbf{C}_n)$ with $e \neq \mathrm{id}_{C_n}$. We must have $e(c_{n-1}) = 1$. So $e(y) \neq y$ and therefore $\alpha(e(y)) = 1 = e(c_{n-1}) = e(\alpha(y))$. For all $x \in X \backslash \{y\}$, we have $e(x) \geqslant x > y$ and hence $\alpha(e(x)) = 1 = e(1) = e(\alpha(x))$. Thus $\alpha\colon \mathbb{X} \to \mathbb{C}_n$ is a morphism.

It remains to check that α does not extend to a term function of $\mathbf{C}_n$. Since $3 < n-1$, it is easy to check that the binary relation

$$r := \{(0,0), (c_2, c_3), (c_{n-1}, c_{n-1}), (1,1)\}$$

forms a subalgebra $\mathbf{r}$ of $\mathbf{C}_n^2$. We have

$$y = (c_2, c_{n-1}) \in X \quad \text{and} \quad z := (c_3, c_{n-1}) \in X,$$

with $(y, z) \in r^X$ but $(\alpha(y), \alpha(z)) = (c_{n-1}, 1) \notin r$. So α does not preserve r. Thus α cannot extend to a term function of $\mathbf{C}_n$, as $\mathbf{r} \leqslant \mathbf{C}_n^2$. □

Lemma 2.1.13. *For $n \leqslant 4$, the n-element Heyting chain is strictly endoprimal.*

Proof. The Heyting chains $\mathbf{C}_1$ and $\mathbf{C}_2$ are term-equivalent to Boolean algebras, and therefore they are strictly endoprimal by Example 2.1.5.

Now consider the 3-element Heyting chain $\mathbf{C}_3 = \langle \{0, c, 1\}; \vee, \wedge, \to, 0, 1 \rangle$. Note that $\mathbf{C}_3$ has a unique non-identity endomorphism e, given by $e(0) = 0$ and $e(c) = e(1) = 1$, and a unique proper partial endomorphism $p := \mathrm{id}_{\{0,1\}}$. To see that $\mathbf{C}_3$ is strictly endoprimal, define $\mathbb{C}_3 := \langle C_3; \mathrm{End}(\mathbf{C}_3) \rangle$ and let $\alpha\colon \mathbb{X} \to \mathbb{C}_3$ be a morphism, where $\mathbb{X} \leqslant \mathbb{C}_3^k$ for some $k \geqslant 1$. Clearly X is closed under p and, since $\{0, 1\} = \text{fix}(e)$, the map α preserves p. As $\mathbf{C}_3$ is strictly p-endoprimal, by Lemma 2.1.9, it follows that α extends to a k-ary term function of $\mathbf{C}_3$. Thus $\mathbf{C}_3$ is strictly endoprimal.

Finally we consider the Heyting chain $\mathbf{C}_4 = \langle \{0, c, d, 1\}; \vee, \wedge, \rightarrow, 0, 1 \rangle$, where $0 < c < d < 1$. The five maximal partial endomorphisms of $\mathbf{C}_4$ are given in the table in Figure 2.2. Note that the proper partial endomorphism h does not extend to an endomorphism.

	id	e	f	g	h
1	1	1	1	1	1
d	d	1	1	1	c
c	c	c	d	1	·
0	0	0	0	0	0

Figure 2.2: Maximal partial endomorphisms of $\mathbf{C}_4$

We shall use Lemma 1.2.11. We want to show that every compatible relation on $\mathbf{C}_4$ is conjunct-atomic definable from $\text{End}(\mathbf{C}_4)$. We know that $\mathbf{C}_4$ is strictly p-endoprimal, by Lemma 2.1.9. So every compatible relation on $\mathbf{C}_4$ is conjunct-atomic definable from binary relations of the form

$$\ker(p, q) := \{ (a, b) \in C_4^2 \mid p(a) = q(b) \},$$

where $p, q \in \text{End}_{\text{p}}(\mathbf{C}_4)$. It now suffices to show that each such relation $\ker(p, q)$ is conjunct-atomic definable from $\text{End}(\mathbf{C}_4)$. Up to symmetry, there are two cases to consider.

Case 1: $p \neq h$ and $q \neq h$. Since h is the only partial endomorphism of $\mathbf{C}_4$ that does not extend to an endomorphism, we can choose $\overline{p}$ and $\overline{q}$ in $\text{End}(\mathbf{C}_4)$ extending p and q, respectively. The unary relations $\text{dom}(p)$ and $\text{dom}(q)$ on $\mathbf{C}_4$ are conjunct-atomic definable from $\text{End}(\mathbf{C}_4)$, as

$$\{0, 1\} = \text{fix}(f), \quad \{0, c, 1\} = \text{fix}(e) \quad \text{and} \quad \{0, d, 1\} = \text{eq}(e, f).$$

Since $\ker(p, q) = \ker(\overline{p}, \overline{q}) \cap (\text{dom}(p) \times \text{dom}(q))$, it follows that $\ker(p, q)$ is conjunct-atomic definable from $\mathrm{End}(\mathbf{C}_4)$.

Case 2: $p = h$. As $h^{-1} \circ q \in \mathrm{End}_{\mathrm{p}}(\mathbf{C}_4)\backslash\{h\}$ and $\ker(h, q) = \ker(\mathrm{id}, h^{-1} \circ q)$, this case is covered by Case 1. □

Theorem 2.1.14. *A finite relative Stone algebra is strictly endoprimal if and only if it belongs to* $\mathsf{ISP}(\mathbf{C}_4)$.

Proof. Let $\mathbf{A}$ be a finite relative Stone algebra. Then there exists $n \geqslant 1$ such that $\mathsf{ISP}(\mathbf{A}) = \mathsf{ISP}(\mathbf{C}_n)$, by Lemma 2.1.8. By Lemmas 2.1.4, 2.1.12 and 2.1.13, the algebra $\mathbf{A}$ is strictly endoprimal if and only if $n \leqslant 4$. For all $k, \ell \geqslant 1$, we have $\mathbf{C}_k \in \mathsf{ISP}(\mathbf{C}_\ell)$ if and only if $k \leqslant \ell$. So the result follows. □

The following example shows that strict p-endoprimality, by itself, is not sufficient to guarantee that a finite algebra $\mathbf{M}$ admits only finitely many relations. In the next section we introduce a condition on the partial unary algebra $\mathbb{M} := \langle M; \mathrm{End}_{\mathrm{p}}(\mathbf{M})\rangle$ that, together with strict p-endoprimality, is sufficient.

Example 2.1.15. *The Heyting algebra* $\mathbf{2}^2 \oplus \mathbf{1}$ *is strictly endoprimal but admits infinitely many relations.*

Proof. The Heyting algebra $\mathbf{A} := \mathbf{2}^2 \oplus \mathbf{1}$ is strictly p-endoprimal by Lemma 2.1.7. It is easy to check that each partial endomorphism of $\mathbf{A}$ extends to an endomorphism of $\mathbf{A}$, and that the domain of each partial endomorphism of $\mathbf{A}$ is conjunct-atomic definable from $\mathrm{End}(\mathbf{A})$. Thus it follows (using Definition 2.1.1) that $\mathbf{A}$ is strictly endoprimal. But any finite Heyting algebra that is not a chain admits infinitely many relations [24, 3.4]. □

Remark 2.1.16. Some of the examples from this section can also be obtained via natural duality theory, since ‘strictly endoprimal’ is weaker than ‘strongly endodualisable’, and ‘strictly p-endoprimal’ is weaker than ‘strongly dualisable via partial endomorphisms’.

The Heyting algebras $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{2}^2 \oplus \mathbf{1}$ are all strongly endodualisable; see [13, 4.2.3]. But $\mathbf{C}_4$ is strictly endoprimal without being strongly endodualisable. (A strongly endodualisable algebra must be injective in the quasi-variety it generates [13, 6.1.2].) While all finite Boolean algebras are strictly endoprimal, only the 1-element and 2-element ones are strongly endodualisable. (A non-trivial finite algebra that is strongly dualisable via partial endomorphisms must be subdirectly irreducible [13, 6.2.2].)

Every finite Heyting chain is strongly dualisable via partial endomorphisms; see [13, 4.2.2]. But this is not true in general for finite relative Stone algebras (see [13, 6.2.2]), even though they are all strictly p-endoprimal.

Remark 2.1.17. By Lemma 2.1.9, for all $n \in \mathbb{N}$, the Heyting chain $\mathbf{C}_n$ is strictly p-endoprimal, that is, $\mathbb{C}_n = \langle C_n; \mathrm{End}_{\mathrm{p}}(\mathbf{C}_n)\rangle$ satisfies (IC). Therefore every compatible relation on $\mathbf{C}_n$ is equivalent to one of the form $\mathrm{E}(\mathbb{X}){\restriction}_S$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{C}_n)$ and S is a non-empty generating set for $\mathbb{X}$. For $n \leqslant 4$, the axiomatisations for finite structures $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{C}_n)$ are known; see Davey and Talukder [29] and [30]. In fact, in these cases, the axiomatisations are known for the topological structures in the class $\mathfrak{X}_n := \mathsf{IS^0_cP^+}(\mathbb{C}_n^{\mathcal{T}})$, where $\mathbb{C}_n^{\mathcal{T}}$ is the topological structure obtained from $\mathbb{C}_n$ by adding the discrete topology. So far, axiomatisations for the class $\mathfrak{X}_n$, with $n \geqslant 5$, are not known. In the next sections, without axiomatisations, we introduce a special property of the structures in $\mathsf{ISP}(\mathbb{C}_n)$ and use it to encode compatible relations on $\mathbf{C}_n$.

2.2 Linear partial unary algebras

In this section, we generalise the property 'linear' from unary algebras to partial unary algebras.

Definition 2.2.1. Let $\mathbb{M} = \langle M; H\rangle$ be a partial unary algebra (so H is a set of unary partial operations on M). We say $\mathbb{M}$ is **linear** if, for all unary term functions u and v of $\mathbb{M}$, there exists a unary term function w of $\mathbb{M}$ such that

$$u{\restriction}_D = w \circ v{\restriction}_D \quad \text{or} \quad v{\restriction}_D = w \circ u{\restriction}_D, \quad \text{where } D := \mathrm{dom}(u) \cap \mathrm{dom}(v).$$

(Recall that the term functions are allowed to be partial operations.)

As is the case for unary algebras, the linearity of a partial unary algebra $\mathbb{M}$ implies that the overall structure of each partial algebra in $\mathsf{ISP}(\mathbb{M})$ is quite simple. To see this, we first need to introduce some notation.

Let $\mathbb{A}$ be a partial unary algebra. For each $a \in A$, we use $\mathrm{sg}_{\mathbb{A}}(a)$ to denote the subuniverse of $\mathbb{A}$ generated by a. Define the ordered set $\mathbf{Sub}_1(\mathbb{A})$ to be

$$\mathrm{Sub}_1(\mathbb{A}) := \{\, \mathrm{sg}_{\mathbb{A}}(a) \mid a \in A \,\} \cup \{\varnothing\}$$

under set-inclusion. Recall that an ordered set $\mathbf{S} = \langle S; \leqslant \rangle$ is a **tree** if it is connected and the ordered set $\downarrow_{\mathbf{S}}(s) := \{\, t \in S \mid t \leqslant s \,\}$ is a chain, for all $s \in S$.

Example 2.2.2. Let $\mathbf{C}_4 = \langle \{0, c, d, 1\}; \vee, \wedge, \to, 0, 1 \rangle$ be the four-element Heyting chain, where $0 < c < d < 1$. Then $\mathbb{C}_4 = \langle \{0, c, d, 1\}; f, h \rangle$ is an alter ego of $\mathbf{C}_4$, where f and h are the partial endomorphisms given in Figure 2.2. (Note that $e = h \circ f$ and $g = f \circ f$.) Consider the structure $\mathbb{X} \in \mathsf{ISP}_{\mathsf{f}}(\mathbb{C}_4)$ given in Figure 2.3. Define $A_i = \mathrm{sg}_{\mathbb{X}}(i)$, for $i \in X$. Then the ordered set $\mathbf{Sub}_1(\mathbb{X}) = \langle \{\, A_i \mid i \in X \,\} \cup \{\varnothing\}; \subseteq \rangle$ is a tree.

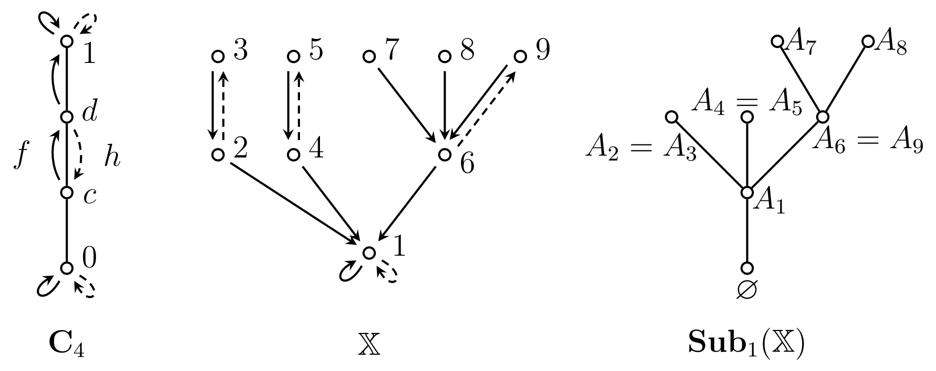


Figure 2.3: An example of $\mathbf{Sub}_1(\mathbb{X})$

The proof of the following lemma is slightly more complicated than that of the analogous lemma for unary algebras [54, 7.2.3], because we cannot make use of free algebras.

Lemma 2.2.3. *Let* $\mathbb{M}$ *be a partial unary algebra. Then the following are equivalent:*

(1) $\mathbb{M}$ *is linear;*

(2) $\mathbf{Sub}_1(\mathbb{A})$ *is a tree, for all* $\mathbb{A} \in \mathsf{ISP}(\mathbb{M})$*;*

(3) $\mathbf{Sub}_1(\mathbb{A})$ *is a chain, for each one-generated partial algebra* $\mathbb{A} \in \mathsf{ISP}(\mathbb{M})$*.*

Proof. (2) $\Rightarrow$ (3): Assume that (2) holds and let $\mathbb{A}$ be a one-generated partial algebra in $\mathsf{ISP}(\mathbb{M})$. Then $\mathbf{Sub}_1(\mathbb{A})$ is a tree. Since $\mathbf{Sub}_1(\mathbb{A})$ has greatest element A (as $\mathbb{A}$ is one-generated), it follows that $\mathbf{Sub}_1(\mathbb{A})$ is a chain.

(3) $\Rightarrow$ (2): Assume that (3) holds and let $\mathbb{A} \in \mathsf{ISP}(\mathbb{M})$. Clearly $\mathbf{Sub}_1(\mathbb{A})$ is connected, as it has least element $\varnothing$. Now let $b \in A$ and define $\mathbb{B} := \mathbf{sg}_{\mathbb{A}}(b)$. Then $\mathbf{Sub}_1(\mathbb{B})$ is a chain, by (3), and we have

$$\downarrow_{\mathbf{Sub}_1(\mathbb{A})}(\mathrm{sg}_{\mathbb{A}}(b)) = \{\, \mathrm{sg}_{\mathbb{A}}(a) \mid a \in B \,\} \cup \{\varnothing\} = \mathrm{Sub}_1(\mathbb{B}).$$

Hence $\downarrow_{\mathbf{Sub}_1(\mathbb{A})}(\mathrm{sg}_{\mathbb{A}}(b))$ is a chain for all $b \in A$. Thus $\mathbf{Sub}_1(\mathbb{A})$ is a tree.

(1) $\Rightarrow$ (3): Assume that $\mathbb{M}$ is linear and let $\mathbb{A}$ be a one-generated partial algebra in $\mathsf{ISP}(\mathbb{M})$. Say that $\mathbb{A} \leqslant \mathbb{M}^I$ and $A = \mathrm{sg}_{\mathbb{A}}(a)$. Now let $b, c \in A$. Then there are unary terms u and v such that $b = u^{\mathbb{A}}(a)$ and $c = v^{\mathbb{A}}(a)$. As $\mathbb{M}$ is linear there is a unary term w such that $u^{\mathbb{M}}{\restriction}_D = w^{\mathbb{M}} \circ v^{\mathbb{M}}{\restriction}_D$ or $v^{\mathbb{M}}{\restriction}_D = w^{\mathbb{M}} \circ u^{\mathbb{M}}{\restriction}_D$, where $D := \mathrm{dom}(u^{\mathbb{M}}) \cap \mathrm{dom}(v^{\mathbb{M}})$. We have $a \in \mathrm{dom}(u^{\mathbb{A}}) \cap \mathrm{dom}(v^{\mathbb{A}})$ and $\mathbb{A} \leqslant \mathbb{M}^I$. So $a(i) \in D$, for all $i \in I$, and therefore $b = u^{\mathbb{A}}(a) = w^{\mathbb{A}}(v^{\mathbb{A}}(a)) = w^{\mathbb{A}}(c)$ or $c = v^{\mathbb{A}}(a) = w^{\mathbb{A}}(u^{\mathbb{A}}(a)) = w^{\mathbb{A}}(b)$. This implies that $\mathrm{sg}_{\mathbb{A}}(b) \subseteq \mathrm{sg}_{\mathbb{A}}(c)$ or $\mathrm{sg}_{\mathbb{A}}(c) \subseteq \mathrm{sg}_{\mathbb{A}}(b)$. Hence $\mathbf{Sub}_1(\mathbb{A})$ is a chain.

(3) $\Rightarrow$ (1): Assume that (3) holds. To see that $\mathbb{M}$ is linear, let u and v be unary term functions of $\mathbb{M}$. Define $D := \mathrm{dom}(u) \cap \mathrm{dom}(v)$. We may assume that $D \neq \varnothing$. Let $\mathrm{id}_D \colon D \to M$ be the inclusion map and define $\mathbb{A} := \mathbf{sg}_{\mathbb{M}^D}(\mathrm{id}_D)$. Then $\mathbf{Sub}_1(\mathbb{A})$ is a chain, by (3). We have $u{\restriction}_D = u^{\mathbb{M}^D}(\mathrm{id}_D) \in A$ and $v{\restriction}_D \in A$, whence $\mathrm{sg}_{\mathbb{A}}(u{\restriction}_D) \subseteq \mathrm{sg}_{\mathbb{A}}(v{\restriction}_D)$ or $\mathrm{sg}_{\mathbb{A}}(v{\restriction}_D) \subseteq \mathrm{sg}_{\mathbb{A}}(u{\restriction}_D)$. So there is a unary term w such that $u{\restriction}_D = w^{\mathbb{A}}(v{\restriction}_D)$ or $v{\restriction}_D = w^{\mathbb{A}}(u{\restriction}_D)$. It follows that $u{\restriction}_D = w^{\mathbb{M}} \circ v{\restriction}_D$ or $v{\restriction}_D = w^{\mathbb{M}} \circ u{\restriction}_D$, as $\mathbb{A} \leqslant \mathbb{M}^D$. Thus $\mathbb{M}$ is linear. ☐

In the next section, we shall prove that a finite strictly p-endoprimal algebra $\mathbf{M}$ admits only finitely many relations provided the partial unary algebra $\mathbb{M} := \langle M; \mathrm{End_p}(\mathbf{M})\rangle$ is linear. Using Lemma 1.2.10, we know that every compatible relation on $\mathbf{M}$ is equivalent to a relation of the form $\mathrm{E}(\mathbb{X}){\restriction_S}$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a generating set for $\mathbb{X}$. We shall view such a pair $(\mathbb{X}; S)$ as the structure on the set X obtained by enriching the partial unary algebra $\mathbb{X}$ with the unary relation S. So, in particular, an isomorphism $\psi\colon (\mathbb{X}; S) \to (\mathbb{Y}; T)$ is an isomorphism $\psi\colon \mathbb{X} \to \mathbb{Y}$ such that $\psi(S) = T$.

The proof that the algebra $\mathbf{M}$ admits only finitely many relations is in two parts:

1. each compatible relation on $\mathbf{M}$ is equivalent to a relation of the form $\mathrm{E}(\mathbb{X}){\restriction_S}$, where $(\mathbb{X}; S)$ is a 'pruned' structure;
2. up to isomorphism, there are only finitely many such structures.

We finish this section by defining pruned structures.

Definition 2.2.4. Let $\mathbb{M}$ be a finite partial unary algebra and assume that $\mathbb{M}$ is linear. Let $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and let S be a generating set for $\mathbb{X}$. Then $\mathbf{Sub}_1(\mathbb{X})$ is a finite tree, by Lemma 2.2.3, since $\mathbb{M}$ is linear. Now let $A \in \mathrm{Sub}_1(\mathbb{X})$ and let $\mathcal{U}_A$ denote the set of all upper covers of A in $\mathbf{Sub}_1(\mathbb{X})$. For each $B \in \mathcal{U}_A$, define the sets

$$B^{\vartriangle} := \{\, x \in X \mid B \subseteq \mathrm{sg}_{\mathbb{X}}(x) \,\} \quad \text{and} \quad B^{\blacktriangle} := \mathrm{sg}_{\mathbb{X}}(B^{\vartriangle}).$$

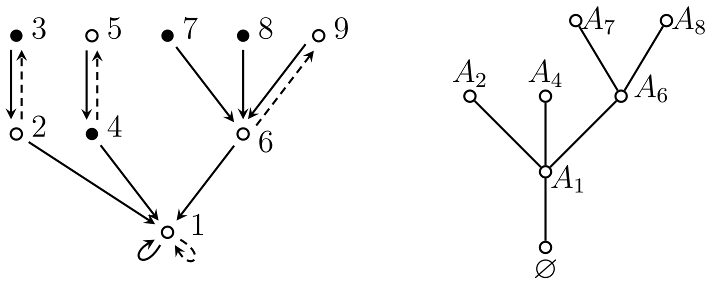
It is easy to check that $B^{\blacktriangle} = A \cup B^{\vartriangle}$ (see [54, 7.2.7]) and that $A \cap B^{\vartriangle} = \varnothing$.

(i) For each $B \in \mathcal{U}_A$, we shall call $B^{\blacktriangle}$ a **branch** of $(\mathbb{X}; S)$ up from A.

(ii) Two branches $B_1^{\blacktriangle}$ and $B_2^{\blacktriangle}$ of $(\mathbb{X}; S)$ up from A are **alike** if there is an isomorphism $\psi\colon (\mathbb{B}_1^{\blacktriangle}; S_1) \to (\mathbb{B}_2^{\blacktriangle}; S_2)$ that fixes each element of A, where $S_i := S \cap B_i^{\blacktriangle}$, for $i \in \{1, 2\}$.

(iii) We shall say that the structure $(\mathbb{X}; S)$ is **pruned** if, for each $A \in \mathrm{Sub}_1(\mathbb{X})$, there do not exist three distinct branches of $(\mathbb{X}; S)$ up from A that are pairwise alike.

Example 2.2.5. We continue Example 2.2.2. The set $S := \{3, 4, 7, 8\}$ is a generating set for $\mathbb{X}$; see Figure 2.4. Now we look at branches of $(\mathbb{X}; S)$. We have $\mathcal{U}_{A_1} = \{A_2, A_4, A_6\}$ and hence altogether there are three branches of $(\mathbb{X}; S)$ up from A_1. They are $A_2^{\blacktriangle} = \{1, 2, 3\}$, $A_4^{\blacktriangle} = \{1, 4, 5\}$ and $A_6^{\blacktriangle} = \{1, 6, 7, 8, 9\}$. It is easy to see from Figure 2.4 that no two of these branches are alike. Similarly, there are only two branches of $(\mathbb{X}; S)$ up from A_6 and they are $A_7^{\blacktriangle} = \{1, 6, 7, 9\}$ and $A_8^{\blacktriangle} = \{1, 6, 8, 9\}$. The two branches of $(\mathbb{X}; S)$ up from A_6 are alike. It follows that $(\mathbb{X}; S)$ is a pruned structure.



$(\mathbb{X}; S)$ $\mathbf{Sub}_1(\mathbb{X})$

Figure 2.4: An example of a pruned structure

Lemma 2.2.6. *Let* $\mathbb{M}$ *be a finite partial unary algebra and assume that* $\mathbb{M}$ *is linear. Let* $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ *with generating set* S*. Let* $A \in \mathrm{Sub}_1(\mathbb{X})$ *and let* B *be an upper cover of* A *in* $\mathbf{Sub}_1(\mathbb{X})$*. Then* $S \cap B^{\blacktriangle}$ *is a generating set for the branch* $\mathbb{B}^{\blacktriangle}$ *of* $(\mathbb{X}; S)$*.*

Proof. We want to show that $B^{\blacktriangle} = \mathrm{sg}_{\mathbb{X}}(S \cap B^{\blacktriangle})$. Since $B^{\blacktriangle} = \mathrm{sg}_{\mathbb{X}}(B^{\vartriangle})$, it suffices to check that $B^{\vartriangle} \subseteq \mathrm{sg}_{\mathbb{X}}(S \cap B^{\blacktriangle})$. So let $b \in B^{\vartriangle}$. Since S is a generating set for the partial unary algebra $\mathbb{X}$, there exist $s \in S$ and a unary term function f of $\mathbb{X}$ such that $b = f(s)$. Suppose $s \in X \setminus B^{\vartriangle}$. It is easy to check that $X \setminus B^{\vartriangle}$ is a subuniverse of $\mathbb{X}$ (see [54, 7.2.7]). So $f(s) \in X \setminus B^{\vartriangle}$. But we have $f(s) = b \in B^{\vartriangle}$, which is a contradiction. Hence $s \in B^{\vartriangle}$, and so $s \in S \cap B^{\vartriangle} \subseteq S \cap B^{\blacktriangle}$. Thus $b = f(s) \in \mathrm{sg}_{\mathbb{X}}(S \cap B^{\blacktriangle})$, as required. □

2.3 A general sufficient condition

In this section, we give a sufficient condition for a finite algebra to admit only finitely many compatible relations and show that every finite Heyting chain satisfies this condition. To do this, we first prove several lemmas.

Lemma 2.3.1. *Let* $\mathbf{M}$ *be a finite strictly p-endoprimal algebra and assume that the partial unary algebra* $\mathbb{M} := \langle M; \mathrm{End}_{\mathrm{p}}(\mathbf{M})\rangle$ *is linear. Then each compatible relation on* $\mathbf{M}$ *is equivalent to one of the form* $\mathrm{E}(\mathbb{Y}){\restriction}_T$*, for some* $\mathbb{Y} \in \mathsf{ISP_f}(\mathbb{M})$ *with generating set* T *such that* $(\mathbb{Y}; T)$ *is pruned.*

Proof. Let r be a compatible relation on $\mathbf{M}$. As $\mathbf{M}$ is strictly p-endoprimal, by Lemma 1.2.10 the relation r is equivalent to $\mathrm{E}(\mathbb{X}){\restriction}_S$, for some $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and generating set S for $\mathbb{X}$. Assume that the structure $(\mathbb{X}; S)$ is not pruned. We shall construct $\mathbb{Y} \lneq \mathbb{X}$ such that the relation r is equivalent to $\mathrm{E}(\mathbb{Y}){\restriction}_T$, where T is a generating set for $\mathbb{Y}$. Since $\mathbb{X}$ is finite, we can continue this process until we obtain a pruned structure.

Since $(\mathbb{X}; S)$ is not pruned, there are three distinct branches $B_1^{\blacktriangle}, B_2^{\blacktriangle}, B_3^{\blacktriangle}$ of $(\mathbb{X}; S)$ up from some $A \in \mathrm{Sub}_1(\mathbb{X})$ that are all alike. Define $S_i := S \cap B_i^{\blacktriangle}$. Then there are isomorphisms $\psi_{ij} \colon (\mathbb{B}_i^{\blacktriangle}; S_i) \to (\mathbb{B}_j^{\blacktriangle}; S_j)$ such that $\psi_{ij}{\restriction}_A = \mathrm{id}_A$, for $i, j \in \{1, 2, 3\}$. Now define $Y := X \setminus B_3^{\vartriangle}$ and $T := S \cap Y$. Then $\mathbb{Y} \lneq \mathbb{X}$. It remains to check that T is a generating set for $\mathbb{Y}$ and that the two relations $\mathrm{E}(\mathbb{X}){\restriction}_S$ and $\mathrm{E}(\mathbb{Y}){\restriction}_T$ on $\mathbf{M}$ are equivalent.

Recall that we have $B_i^{\blacktriangle} = A \cup B_i^{\vartriangle}$ and $A \cap B_i^{\vartriangle} = \varnothing$, for each $i \in \{1, 2, 3\}$. The three subsets $B_1^{\vartriangle}, B_2^{\vartriangle}, B_3^{\vartriangle}$ of X are disjoint, so we also have $B_1^{\blacktriangle} \subseteq Y$ and $B_2^{\blacktriangle} \subseteq Y$.

Claim 1: T is a generating set for $\mathbb{Y}$.

Let $y \in Y = X \backslash B_3^{\vartriangle}$. Since S is a generating set for the partial unary algebra $\mathbb{X}$, there is some $s \in S$ such that $y \in \mathrm{sg}_{\mathbb{X}}(s)$. We can assume that $s \notin T$. So $s \in B_3^{\vartriangle}$ and therefore $y \in B_3^{\blacktriangle} \backslash B_3^{\vartriangle} = A \subseteq B_1^{\blacktriangle}$. Thus $y \in \mathrm{sg}_{\mathbb{X}}(t)$, for some $t \in S \cap B_1^{\blacktriangle}$, by Lemma 2.2.6. But $B_1^{\blacktriangle} \subseteq Y$ and therefore $t \in T$.

Claim 2: $\mathrm{E}(\mathbb{Y})\restriction_T$ *is conjunct-atomic definable from* $\mathrm{E}(\mathbb{X})\restriction_S$.

We use Lemma 1.2.13. Define a map $\rho\colon X \to Y$ by

$$\rho(x) = \begin{cases} \psi_{32}(x), & \text{if } x \in B_3^{\vartriangle}, \\ x, & \text{otherwise.} \end{cases}$$

We can see that $\rho\restriction_{B_3^{\blacktriangle}} = \psi_{32}$ (as $B_3^{\blacktriangle} = A \cup B_3^{\vartriangle}$ and ψ_{32} fixes each element of A) and $\rho\restriction_Y = \mathrm{id}_Y$. Since $\psi_{32}\colon \mathbb{B}_3^{\blacktriangle} \to \mathbb{B}_2^{\blacktriangle} \leqslant \mathbb{Y}$ and $\mathrm{id}_Y\colon \mathbb{Y} \to \mathbb{Y}$ are morphisms and $B_3^{\blacktriangle} \cup Y = X$, it follows that $\rho\colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ is a morphism. Therefore $\rho\colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ is a retraction with $\rho\restriction_Y = \mathrm{id}_Y$. Since $\psi_{32}(S_3) = S_2 \subseteq T$, we have $\rho(S) = T \subseteq S$. Hence $\mathrm{E}(\mathbb{Y})\restriction_T$ is conjunct-atomic definable from $\mathrm{E}(\mathbb{X})\restriction_S$, by Lemma 1.2.13.

Claim 3: $\mathrm{E}(\mathbb{X})\restriction_S$ *is conjunct-atomic definable from* $\mathrm{E}(\mathbb{Y})\restriction_T$.

We shall use Lemma 1.2.12. Let $\varphi\colon S \to M$ be a map that does not extend to a morphism from $\mathbb{X}$ to $\mathbb{M}$. Then since $X = Y \cup B_3^{\blacktriangle}$ and $A = Y \cap B_3^{\blacktriangle}$, one of the following three conditions holds:

(a) $\varphi\restriction_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$;

(b) $\varphi\restriction_{S_3}\colon S_3 \to M$ does not extend to a morphism from $\mathbb{B}_3^{\blacktriangle}$ to $\mathbb{M}$;

(c) there are morphisms $\mu\colon \mathbb{Y} \to \mathbb{M}$ and $\nu\colon \mathbb{B}_3^{\blacktriangle} \to \mathbb{M}$ extending $\varphi\restriction_T$ and $\varphi\restriction_{S_3}$ respectively, but $\mu\restriction_A \neq \nu\restriction_A$.

We want to find a morphism $\omega\colon (\mathbb{Y}; T) \to (\mathbb{X}; S)$ such that $\varphi \circ \omega\restriction_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$.

Case (a): $\varphi\restriction_T\colon T \to M$ *does not extend to a morphism from* $\mathbb{Y}$ *to* $\mathbb{M}$.

Let $\omega\colon \mathbb{Y} \to \mathbb{X}$ be the inclusion. Then we have $\omega(T) = T \subseteq S$ and the map $\varphi \circ \omega\restriction_T = \varphi\restriction_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$, by assumption.

Case (b): *$\varphi{\restriction}_{S_3}\colon S_3 \to M$ does not extend to a morphism from $\mathbb{B}_3^{\blacktriangle}$ to $\mathbb{M}$.*

We can define the morphism $\omega\colon \mathbb{Y} \to \mathbb{X}$ by

$$\omega(y) = \begin{cases} \psi_{23}(y), & \text{if } y \in B_2^{\vartriangle}, \\ y, & \text{otherwise.} \end{cases}$$

Then $\omega(T) \subseteq S$. The map $\varphi \circ \omega{\restriction}_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$, because $\psi_{23}\colon \mathbb{B}_2^{\blacktriangle} \to \mathbb{B}_3^{\blacktriangle}$ is an isomorphism, and therefore the map $\varphi \circ \omega{\restriction}_{S_2} = \varphi{\restriction}_{S_3} \circ \psi_{23}{\restriction}_{S_2}$ does not extend to a morphism from $\mathbb{B}_2^{\blacktriangle}$ to $\mathbb{M}$.

Case (c): *There are morphisms $\mu\colon \mathbb{Y} \to \mathbb{M}$ and $\nu\colon \mathbb{B}_3^{\blacktriangle} \to \mathbb{M}$ extending $\varphi{\restriction}_T$ and $\varphi{\restriction}_{S_3}$ respectively, but $\mu{\restriction}_A \neq \nu{\restriction}_A$.*

As in Case (b), we can define the morphism $\omega\colon \mathbb{Y} \to \mathbb{X}$ by

$$\omega(y) = \begin{cases} \psi_{23}(y), & \text{if } y \in B_2^{\vartriangle}, \\ y, & \text{otherwise,} \end{cases}$$

and we have $\omega(T) \subseteq S$. As $\mu{\restriction}_A \neq \nu{\restriction}_A$, then there is some $a \in A$ such that $\mu(a) \neq \nu(a)$. For each $i \in \{1, 2\}$, since $S_i = S \cap B_i^{\blacktriangle}$ is a generating set for $\mathbb{B}_i^{\blacktriangle}$ (by Lemma 2.2.6) it follows that $a = f_i^{\mathbb{X}}(b_i)$, for some $f_i \in \mathrm{End_p}(\mathbf{M})$ and $b_i \in S_i$. Define $b_3 := \omega(b_2) \in S_3$. Then

$$a = \omega(a) = \omega(f_2(b_2)) = f_2(\omega(b_2)) = f_2(b_3).$$

Since $\mu\colon \mathbb{Y} \to \mathbb{M}$ and $\nu\colon \mathbb{B}_3^{\blacktriangle} \to \mathbb{M}$ are morphisms extending $\varphi{\restriction}_T$ and $\varphi{\restriction}_{S_3}$, respectively, we have

$$\mu(a) = \mu(f_1(b_1)) = f_1(\mu(b_1)) = f_1(\varphi(b_1)) = f_1\big((\varphi \circ \omega{\restriction}_T)(b_1)\big), \text{ and}$$
$$\nu(a) = \nu(f_2(b_3)) = f_2(\nu(b_3)) = f_2(\varphi(b_3)) = f_2\big(\varphi(\omega(b_2))\big) = f_2\big((\varphi \circ \omega{\restriction}_T)(b_2)\big).$$

As $\mu(a) \neq \nu(a)$, by assumption, this gives $f_1\big((\varphi\circ\omega{\restriction}_T)(b_1)\big) \neq f_2\big((\varphi\circ\omega{\restriction}_T)(b_2)\big)$. But $f_1(b_1) = a = f_2(b_2)$, and so $\varphi \circ \omega{\restriction}_T$ cannot be extended to a morphism from $\mathbb{Y}$ to $\mathbb{M}$.

Hence $\mathrm{E}(\mathbb{X})\restriction_S$ is conjunct-atomic definable from $\mathrm{E}(\mathbb{Y})\restriction_T$, by Lemma 1.2.12.

□

Now we obtain an upper bound on the number of pruned structures in $\mathsf{ISP_f}(\mathbb{M})$. We use the following easy lemma.

Lemma 2.3.2. *Let $\mathbb{M} = \langle M; G, H, R\rangle$ be a finite structure and define $n := |M|$. Then, up to isomorphism, there are at most 2^n one-generated structures in $\mathsf{ISP_f}(\mathbb{M})$, each of which has size at most n^n.*

Proof. Let $\mathbb{X} \leqslant \mathbb{M}^I$ with $X = \mathrm{sg}_{\mathbb{X}}(y)$, for some non-empty set I and some $y \in M^I$. Then $\ker(y) \subseteq \ker(x)$, for all $x \in X \subseteq M^I$. The equivalence relation $\theta = \ker(y)$ on I has at most $|M|$ blocks, and so it follows that $|X| \leqslant |M^M|$.

Define $y' \in M^{I/\theta}$ by $y'(i/\theta) := y(i)$, for all $i \in I$. Then y' is well defined and $y' \in M^{I/\theta}$ is obtained from y via co-ordinate repetition removal.

Let $\mathbb{X}' = \mathbf{sg}_{\mathbb{M}^{I/\theta}}(y')$. Define the map $\alpha\colon M^{I/\theta} \to M^I$ by $\alpha(\varphi)(i) := \varphi(i/\theta)$, for all $\varphi \in M^{I/\theta}$. Then we have $\pi_i \circ \alpha = \pi_{i/\theta}$, for all $i \in I$. Hence α is the product map $\prod_{i\in I} \pi_{i/\theta}$, and so α is a morphism from $\mathbb{M}^{I/\theta}$ to $\mathbb{M}^I$ with $\alpha(y') = y$.

Now let $\varphi, \psi \in M^{I/\theta}$ and assume that $\alpha(\varphi) = \alpha(\psi)$, then

$$\begin{aligned} & \alpha(\varphi)(i) = \alpha(\psi)(i), \text{ for all } i \in I \\ \implies & \varphi(i/\theta) = \psi(i/\theta), \text{ for all } i \in I \\ \implies & \varphi = \psi. \end{aligned}$$

Hence α is one-to-one.

Let $r \in \mathrm{dom}(H) \cup R$ be k-ary and let $\varphi_1, \dots, \varphi_k \in M^{I/\theta}$. Assume that $\big(\alpha(\varphi_1), \dots, \alpha(\varphi_k)\big) \in r^{\mathbb{M}^I}$. Then

$$\begin{aligned} & \big(\alpha(\varphi_1)(i), \dots, \alpha(\varphi_k)(i)\big) \in r^{\mathbb{M}}, \text{for all } i \in I \\ \implies & \big(\varphi_1(i/\theta), \dots, \varphi_k(i/\theta)\big) \in r^{\mathbb{M}}, \text{for all } i \in I \\ \implies & \big(\varphi_1, \dots, \varphi_k\big) \in r^{\mathbb{M}^{I/\theta}}. \end{aligned}$$

Thus α is an embedding, by Lemma 1.1.5.

Since $\alpha(y') = y$, we have $\alpha(X') = \alpha(\mathrm{sg}_{M^I/\theta}(y')) \subseteq \mathrm{sg}_{M^I}(y) = X$. Now let $z \in X = \mathrm{sg}_{\mathbb{X}}(y)$. Then $z = t^{\mathbb{X}}(y)$ for some unary term t. Since α is an embedding and $\alpha(y') = y \in \mathrm{dom}(t^{\mathbb{X}})$ we have $y' \in \mathrm{dom}(t^{\mathbb{X}'})$. Define $z' = t^{\mathbb{X}'}(y')$. Then $\alpha(z') = \alpha(t^{\mathbb{X}'}(y')) = t^{\mathbb{X}}(\alpha(y')) = t^{\mathbb{X}}(y) = z$. Thus $z \in \alpha(X')$. Hence $\alpha(X') = X$. It follows that $\alpha{\restriction}_{X'} \colon \mathbb{X}' \to \mathbb{X}$ is an isomorphism. Thus

$$\mathbf{sg}_{\mathbb{M}^I}(y) = \mathbb{X} \cong \mathbb{X}' = \mathbf{sg}_{\mathbb{M}^I/\theta}(y').$$

So, up to isomorphism, the structure $\mathbb{X} = \mathbf{sg}_{\mathbb{M}^I}(y)$ is determined by the subset $y(I)$ of M. It follows that, up to isomorphism, there are at most $2^{|M|} = 2^n$ different one-generated structures in $\mathsf{ISP_f}(\mathbb{M})$. □

Definition 2.3.3. Let $\mathbb{P} = \langle P; \leqslant \rangle$ be a finite ordered set. An element p of P is called a **node** of $\mathbb{P}$ if it is comparable with every other element of $\mathbb{P}$. If the ordered set $\mathbb{P}$ has a least element, then we can define the **depth** of $\mathbb{P}$ to be the length of the ordered set ${\uparrow_{\mathbb{P}}}(q)$, where q is the greatest node of $\mathbb{P}$. For example, the ordered set $\mathbf{Sub}_1(\mathbb{X})$ shown in Figure 2.4 has greatest node A_1 and has depth 2.

Lemma 2.3.4. *Let $\mathbb{M}$ be a finite partial unary algebra and assume that $\mathbb{M}$ is linear. Up to isomorphism, there are only finitely many pruned structures $(\mathbb{X}; S)$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a generating set for $\mathbb{X}$.*

Proof. We define the **depth** of the structure $(\mathbb{X}; S)$ to be the depth of the ordered set $\mathbf{Sub}_1(\mathbb{X})$. We shall argue by induction on depth.

First assume that $(\mathbb{X}; S)$ has depth 0, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a generating set for $\mathbb{X}$. Then $\mathbf{Sub}_1(\mathbb{X})$ is a tree, by Lemma 2.2.3, since $\mathbb{M}$ is linear. So $\mathbf{Sub}_1(\mathbb{X})$ must be a chain, as it has depth 0. Therefore $\mathbb{X}$ is one-generated. It follows from Lemma 2.3.2 that, up to isomorphism, there are at most $2^n \times 2^{n^n}$ pruned structures of depth 0, where $n := |M|$.

Let $k \geqslant 0$. Assume that, up to isomorphism, there is a finite number ℓ of pruned structures of depth at most k. Then there is a finite bound m on the size of each such structure. We now prove that, up to isomorphism, there are only finitely many pruned structures of depth $k+1$.

Let $(\mathbb{X}; S)$ be a pruned structure of depth $k+1$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a generating set for $\mathbb{X}$. Then $\mathbf{Sub}_1(\mathbb{X})$ is a tree, by Lemma 2.2.3. Let A denote the greatest node of $\mathbf{Sub}_1(\mathbb{X})$ and let $\mathcal{U}_A$ denote the set of all upper covers of A. Since $\mathbf{Sub}_1(\mathbb{X})$ has depth $k+1$, we must have $\mathcal{U}_A \neq \varnothing$.

Claim 1: For each $B \in \mathcal{U}_A$, the branch $(\mathbb{B}^\blacktriangle; S \cap B^\blacktriangle)$ of $(\mathbb{X}; S)$ is a pruned structure of depth at most k.

By Lemma 2.2.6, the set $S \cap B^\blacktriangle$ generates $\mathbb{B}^\blacktriangle$. Since A is the greatest node of $\mathbf{Sub}_1(\mathbb{X})$, the depth of $\mathbf{Sub}_1(\mathbb{B}^\blacktriangle)$ is strictly less than the depth of $\mathbf{Sub}_1(\mathbb{X})$.

Now suppose that $(\mathbb{B}^\blacktriangle; S \cap B^\blacktriangle)$ is not a pruned structure. Then there are three distinct branches all alike up from some element C of $\mathrm{Sub}_1(\mathbb{B}^\blacktriangle)$. By the construction of $\mathbb{B}^\blacktriangle$, this yields three distinct branches of $(\mathbb{X}; S)$ all alike up from C, which is a contradiction as $(\mathbb{X}; S)$ is a pruned structure. Hence $(\mathbb{B}^\blacktriangle; S \cap B^\blacktriangle)$ is a pruned structure of depth at most k.

Claim 2: For each pruned structure $(\mathbb{P}; R)$ of depth at most k, there are at most $2m$ branches of $(\mathbb{X}; S)$ up from A that are isomorphic to $(\mathbb{P}; R)$.

Let $\mathbb{A}$ denote the (possibly empty) subalgebra of $\mathbb{X}$ with universe A. Let $(\mathbb{P}; R)$ be any pruned structure of depth at most k and let $\alpha \colon \mathbb{A} \rightarrowtail \mathbb{P}$ be an embedding. Suppose, by way of contradiction, that there are distinct $B_1, B_2, B_3 \in \mathcal{U}_A$ such that, for each $i \in \{1,2,3\}$, there is an isomorphism $\varphi_i \colon (\mathbb{B}_i^\blacktriangle; S_i) \to (\mathbb{P}; R)$ with $\varphi_i{\restriction}_A = \alpha$, where $S_i := S \cap B_i^\blacktriangle$. (Note that $B_i^\blacktriangle = A \cup B_i^\vartriangle$, for each $i \in \{1,2,3\}$.)

There are isomorphisms $\psi_{ij} := \varphi_j^{-1} \circ \varphi_i \colon (\mathbb{B}_i^\blacktriangle; S_i) \to (\mathbb{B}_j^\blacktriangle; S_j)$, for each $i, j \in \{1,2,3\}$. Since $\varphi_i{\restriction}_A = \alpha = \varphi_j{\restriction}_A$, for each $a \in A$ we have

$$\psi_{ij}(a) = \big(\varphi_j^{-1} \circ \varphi_i\big)(a) = \varphi_j^{-1}\big(\varphi_i(a)\big) = \varphi_j^{-1}\big(\varphi_j(a)\big) = a.$$

Hence $\psi_{ij}{\restriction}_A = \mathrm{id}_A$. Since $B_1, B_2, B_3 \in \mathcal{U}_A$, this implies that $B_1^\blacktriangle, B_2^\blacktriangle$ and $B_3^\blacktriangle$ are three distinct branches of $(\mathbb{X}; S)$ up from A that are pairwise alike. That is a contradiction, as $(\mathbb{X}; S)$ is pruned.

Since $A \in \mathrm{Sub}_1(\mathbb{X})$, the partial algebra $\mathbb{A}$ is either empty or one-generated. So there are at most $|P| \leqslant m$ different embeddings $\alpha\colon \mathbb{A} \rightarrowtail \mathbb{P}$. Thus we have shown that there are at most $2m$ branches of $(\mathbb{X}; S)$ up from A that are isomorphic to $(\mathbb{P}; R)$.

Claim 3: $|X| \leqslant 2m^2\ell$.

We are assuming that, up to isomorphism, there are only ℓ different pruned structures of depth at most k, each of which has size at most m. By Claims 1 and 2, it follows that $|B^\blacktriangle| \leqslant m$, for all $B \in \mathcal{U}_A$, and that $|\mathcal{U}_A| \leqslant 2m\ell$. Since A is a node of $\mathbf{Sub}_1(\mathbb{X})$, we have $X = \bigcup \{\, B^\blacktriangle \mid B \in \mathcal{U}_A \,\}$. Thus

$$|X| \leqslant \sum_{B \in \mathcal{U}_A} |B^\blacktriangle| \leqslant m \times 2m\ell = 2m^2\ell.$$

We have now shown that each pruned structure $(\mathbb{X}; S)$ of depth $k+1$ has size at most $2m^2\ell$. It follows that, up to isomorphism, there are only finitely many pruned structures of depth $k+1$. (Since the partial unary algebra $\mathbb{M}$ is finite, we can assume without loss of generality that it is of finite type.)

The depth of the structures in $\mathsf{ISP_f}(\mathbb{M})$ is bounded above by the maximum size of a one-generated structure in $\mathsf{ISP_f}(\mathbb{M})$. It follows by Lemma 2.3.2 that there is a finite bound n^n on the depth of structures in $\mathsf{ISP_f}(\mathbb{M})$, where $n = |M|$. Hence by induction, up to isomorphism, there are only finitely many pruned structures $(\mathbb{X}; S)$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and S is a generating set for $\mathbb{X}$. □

We now give a general condition that guarantees that a finite algebra admits only finitely many relations.

Theorem 2.3.5. *Let* $\mathbf{M}$ *be a finite strictly p-endoprimal algebra and assume that* $\mathbb{M} := \langle M; \mathrm{End_p}(\mathbf{M})\rangle$ *is linear. Then* $\mathbf{M}$ *admits only finitely many relations.*

Proof. By Lemma 2.3.1, every compatible relation on $\mathbf{M}$ is equivalent to one of the form

$$\mathrm{E}(\mathbb{X}){\restriction_S} = \{\, \alpha{\restriction_S} \mid \alpha\colon \mathbb{X} \to \mathbb{M} \,\} \subseteq M^S,$$

for some $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{M})$ and some generating set S for $\mathbb{X}$ such that $(\mathbb{X}; S)$ is pruned. Note that, if $(\mathbb{Y}; T)$ is isomorphic to $(\mathbb{X}; S)$, then the relations $\mathrm{E}(\mathbb{X})\restriction_S$ and $\mathrm{E}(\mathbb{Y})\restriction_T$ are equivalent. So Lemma 2.3.4 completes the proof. □

We finish this chapter by verifying that the previous theorem applies to the n-element Heyting chain $\mathbf{C}_n$, for each $n \geqslant 1$.

Lemma 2.3.6. *The partial unary algebra* $\mathbb{C}_n = \langle C_n; \mathrm{End_p}(\mathbf{C}_n)\rangle$ *is linear, for all* $n \geqslant 1$*.*

Proof. Recall from Lemma 2.1.11 that a map $p\colon A \to C_n$, with $\{0,1\} \subseteq A \subseteq C_n$, is a partial endomorphism of $\mathbf{C}_n$ if and only if (1): p preserves 0 and 1, (2): p is order-preserving, and (3): p is one-to-one away from 1.

Let u and v be partial endomorphisms of $\mathbf{C}_n$. Define $D := \mathrm{dom}(u) \cap \mathrm{dom}(v)$. Without loss of generality, we may assume that $v^{-1}(1) \cap D \subseteq u^{-1}(1) \cap D$.

Define $A := v(D)$. Then $\{0,1\} \subseteq A \subseteq C_n$. We define $w\colon A \to C_n$ by

$$w(v(d)) := u(d),$$

for all $d \in D$. We shall show that w is a well-defined partial endomorphism of $\mathbf{C}_n$. It will then follow that $\mathbb{C}_n = \langle C_n; \mathrm{End_p}(\mathbf{C}_n)\rangle$ is linear, as the definition of w ensures that $u\restriction_D = w \circ v\restriction_D$.

(0) To see that w is well defined, let $c, d \in D$ such that $v(c) = v(d)$. We need to check that $u(c) = u(d)$. We have $c = d$ or $v(c) = v(d) = 1$, as v is one-to-one away from 1. If $c = d$, then $u(c) = u(d)$. So assume that $v(c) = v(d) = 1$. Then $c, d \in v^{-1}(1)$. Since $v^{-1}(1) \cap D \subseteq u^{-1}(1) \cap D$, we have $c, d \in u^{-1}(1)$. So $u(c) = 1 = u(d)$.

(1) Since u and v preserve 0 and 1, so does w.

(2) Let $c, d \in D$ with $v(c) \leqslant v(d)$. If $v(c) = v(d)$, then $w(v(c)) = w(v(d))$. So assume that $v(c) < v(d)$. Then $c < d$ and therefore $u(c) \leqslant u(d)$, as u and v are order-preserving. Hence $w(v(c)) = u(c) \leqslant u(d) = w(v(d))$. Thus w is order-preserving.

(3) Let $c, d \in D$ such that $w(v(c)) = w(v(d))$, i.e., $u(c) = u(d)$. Then $c = d$ or $u(c) = 1$, as u is one-to-one away from 1. So we have $v(c) = v(d)$ or $w(v(c)) = u(c) = 1$. Hence w is one-to-one away from 1.

Thus w is a partial endomorphism of $\mathbf{C}_n$, as required. □

For all $n \geqslant 1$, the n-element Heyting chain $\mathbf{C}_n$ is strictly p-endoprimal (see Lemma 2.1.9) and the partial unary algebra $\mathbb{C}_n = \langle C_n; \mathrm{End}_{\mathrm{p}}(\mathbf{C}_n)\rangle$ is linear. Therefore, by Theorem 2.3.5, we have the following corollary.

Corollary 2.3.7. *Each finite Heyting chain admits only finitely many relations.*

Since each finite Heyting algebra that is not a chain admits infinitely many relations [24, 3.4], this corollary completes the classification for finite Heyting algebras.

Chapter 3

Dualities for Ockham algebras

In this chapter, we introduce two different duality techniques: the restricted Priestley duality for Ockham algebras and the piggyback duality for the quasi-variety generated by a finite subdirectly irreducible Ockham algebra. We shall use these two technique to encode compatible relations on finite Ockham algebras in the next chapter. In Section 3.1, we give a brief introduction to the restricted Priestley duality for Ockham algebras. The categories of dual spaces of certain subvarieties of the variety of Ockham algebras are also introduced in this section. In Section 3.2, we present in detail the piggyback duality for the quasi-variety generated by a single finite subdirectly irreducible Ockham algebra. Finally, in Section 3.3, we investigate an example that plays crucial role for counting compatible relations on Ockham algebras.

3.1 The restricted Priestley duality

An **Ockham algebra** is an algebra $\mathbf{A} = \langle A; \vee, \wedge, f, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ in which $\mathbf{A}^\flat = \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and f is a unary operation such that $f(0) = 1, f(1) = 0$, and f satisfies the De Morgan laws, that is, $f(a \vee b) = f(a) \wedge f(b)$ and $f(a \wedge b) = f(a) \vee f(b)$, for all $a, b \in A$. Within the study of non-classical logics, Ockham algebras arise naturally from De Morgan algebras by omitting the law of double negation. Ockham algebras were first introduced by Berman [5] in 1977. Two years later, in 1979, in his paper [62], Urquhart named this class of algebras Ockham lattices in honour of the logician William of Ockham, who first enounced the De Morgan laws of logic. For more basic results about Ockham algebras, the reader is referred to the text by Blyth and Varlet [10].

We shall use $\mathbf{\mathfrak{O}}$ to denote the category of Ockham algebras. As an Ockham algebra has a bounded distributive lattice reduct, Priestley duality, given in Example 1.3.3, has been used to obtain a dual category for $\mathbf{\mathfrak{O}}$. The dual category for $\mathbf{\mathfrak{O}}$, the category $\mathbf{\mathfrak{Y}}$ of Ockham spaces, was first given by Urquhart [62]. An **Ockham space** is a topological structure $\mathbb{X} = \langle X; \leqslant, g, \mathcal{T}\rangle$ in which

- $\langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, that is, a compact topological ordered space such that for all $x, y \in X$ with $x \nleqslant y$ there exists a clopen down-set U such that $x \notin U$ and $y \in U$,
- g is a continuous order-reversing self-map on X.

The categories $\mathbf{\mathfrak{O}}$ and $\mathbf{\mathfrak{Y}}$ are dually equivalent. The dual functors $H\colon \mathbf{\mathfrak{O}} \to \mathbf{\mathfrak{Y}}$ and $K\colon \mathbf{\mathfrak{Y}} \to \mathbf{\mathfrak{O}}$ are given by

(1) for $\mathbf{A} \in \mathbf{\mathfrak{O}}$, we have $H(\mathbf{A}) = \langle \mathbf{\mathfrak{D}}(\mathbf{A}^\flat, \mathbf{2}); \leqslant, g, \mathcal{T}\rangle$, where

- $\langle \mathbf{\mathfrak{D}}(\mathbf{A}^\flat, \mathbf{2}); \leqslant, \mathcal{T}\rangle$ is the Priestley space dual to the reduct $\mathbf{A}^\flat$ of $\mathbf{A}$; and
- for $h \in \mathbf{\mathfrak{D}}(\mathbf{A}^\flat, \mathbf{2})$, we have $g(h) = c \circ h \circ f$, where c is the usual Boolean complement on $\{0, 1\}$.

(2) for $\mathbb{X} \in \mathbf{\mathfrak{Y}}$, we have $K(\mathbb{X}) = \langle \mathbf{\mathfrak{P}}(\mathbb{X}, \underset{\sim}{2}); \vee, \wedge, f, 0, 1\rangle$, where

- $\langle \mathbf{\mathfrak{P}}(\mathbb{X}, \underset{\sim}{2}); \vee, \wedge, 0, 1\rangle$ is the bounded distributive lattice dual to the Priestley space $\langle X; \leqslant, \mathcal{T}\rangle$; and
- for $\alpha \in \mathbf{\mathfrak{P}}(\mathbb{X}, \underset{\sim}{2})$, we have $f(\alpha) = c \circ \alpha \circ g$.

(3) for each homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ in the category $\mathbf{\mathfrak{O}}$, the morphism $H(\varphi)\colon H(\mathbf{B}) \to H(\mathbf{A})$ is given by $H(\varphi)(x) := x \circ \varphi$, for all $x \in \mathbf{\mathfrak{D}}(\mathbf{B}^\flat, \mathbf{2})$

and for each morphism $\psi\colon \mathbb{X} \to \mathbb{Y}$ in the category $\mathcal{Y}$, the homomorphism $K(\psi)\colon K(\mathbb{Y}) \to K(\mathbb{X})$ is given by $K(\psi)(\alpha) := \alpha \circ \psi$, for all $\alpha \in \mathcal{P}(\mathbb{Y}, \underline{2})$.

The evaluation maps $e_{\mathbf{A}}\colon \mathbf{A} \to KH(\mathbf{A})$ and $\varepsilon_{\mathbb{X}}\colon \mathbb{X} \to HK(\mathbb{X})$, for $\mathbf{A} \in \mathcal{O}$ and $\mathbb{Y} \in \mathcal{Y}$ are now isomorphisms of the category $\mathcal{O}$ and the category $\mathcal{Y}$, respectively.

Note that we have defined the duals of Ockham algebras using hom-sets. It is also common to define them using prime ideals. The prime-ideal duals are order-theoretically dual to the hom-set duals.

Remark 3.1.1. The well-known varieties of Boolean algebras, Kleene algebras, De Morgan algebras, Stone algebras and MS-algebras are subvarieties of $\mathcal{O}$. Each of these subvarieties is generated by a finite subdirectly irreducible algebra. The properties of the unary operation f and the restricted duals of these subvarieties are described as follows:

- The variety $\mathcal{B}$ of Boolean algebras consists of all Ockham algebras $\mathbf{A}$ satisfying $a \wedge f(a) = 0$ and $a \vee f(a) = 1$, for all $a \in A$. The variety $\mathcal{B}$ is generated by $\mathbf{B}$, where $\mathbf{B}$ is the two-element Boolean algebra. The corresponding dual category $\mathcal{Y}^{\mathcal{B}}$ consists of those Ockham spaces $\mathbb{X}$ in which $g = \mathrm{id}$ (and therefore the order is an antichain).

- The variety $\mathcal{K}$ of Kleene algebras consists of all Ockham algebras $\mathbf{A}$ satisfying $a \wedge f(a) \leqslant b \vee f(b)$, for all $a, b \in A$. The variety $\mathcal{K}$ is generated by $\mathbf{K}$, where $\mathbf{K}$ is given in Figure 3.1 (see [3]). The corresponding dual category $\mathcal{Y}^{\mathcal{K}}$ consists of those Ockham spaces $\mathbb{X}$ in which $g^2 = \mathrm{id}$ and x and $g(x)$ are comparable for all $x \in X$ (see [14]).

- The variety $\mathcal{M}$ of De Morgan algebras consists of all Ockham algebras $\mathbf{A}$ satisfying $f^2(a) = a$, for all $a \in A$. The variety $\mathcal{M}$ is generated by $\mathbf{M}$, where $\mathbf{M}$ is given in Figure 3.1 (see [3]). The corresponding dual category $\mathcal{Y}^{\mathcal{M}}$ consists of those Ockham spaces $\mathbb{X}$ in which $g^2 = \mathrm{id}$ (see [14]).

- The variety $\mathcal{S}$ of Stone algebras consists of all Ockham algebras $\mathbf{A}$ satisfying $a \wedge f(a) = 0$ and $f(a) \vee f^2(a) = 1$, for all $a \in A$. The variety $\mathcal{S}$ is generated by **S**, where **S** is given in Figure 3.1 (see [3]). The corresponding dual category $\mathcal{Y}^{\mathcal{S}}$ consists of those Ockham spaces $\mathbb{X}$ in which g maps each element of X up to the unique maximal element above it (see [16], [17]).

- The variety $\mathcal{MS}$ of MS-algebras consists of all Ockham algebras **A** satisfying $a \leqslant f^2(a)$, for all $a \in A$. The variety $\mathcal{MS}$ was first introduced by Blyth and Varlet (see [7]) as a common generalisation of De Morgan algebras and Stone algebras. The variety $\mathcal{MS}$ is generated by $\mathbf{M}_1$, where $\mathbf{M}_1$ is the Ockham algebra given in Figure 3.1 (see [8]). The corresponding dual category $\mathcal{Y}^{\mathcal{MS}}$ consists of those Ockham spaces $\mathbb{X}$ in which $x \leqslant g^2(x)$, for all $x \in X$ (see [9]).

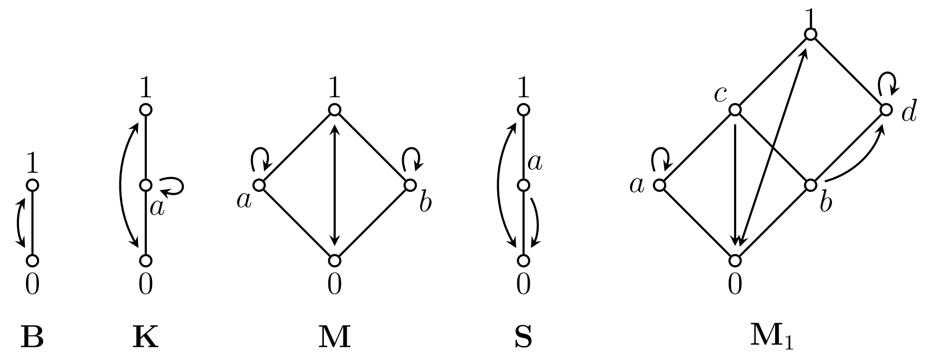


Figure 3.1: Well-known subdirectly irreducible Ockham algebras

3.2 A piggyback duality

The theory of piggyback dualities for varieties generated by a single finite algebra or a finite set of finite algebras with bounded distributive lattice reducts was developed by Davey and Werner [32, 33] and Davey and Priestley [26]. In these papers, the piggyback duality for the quasi-variety generated by a single Ockham algebra is discussed but is not proved in full. We begin this section by introducing the piggyback duality for the quasi-variety generated by a finite algebra with a bounded distributive lattice reduct. We then work out in detail the piggyback duality for the quasi-variety generated by a single finite subdirectly irreducible Ockham algebra, as we require such a duality in the following chapter.

Theorem 3.2.1 (**Piggyback Duality Theorem** [32, 33])**.** *Assume that* $\mathbf{M}$ *is a finite algebra which has a bounded distributive lattice reduct* $\mathbf{M}^\flat = \langle M; \vee, \wedge, 0, 1 \rangle$ *and let* $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$*. Let* Ω_M *be a subset of* $\mathbf{\mathfrak{D}}(\mathbf{M}^\flat, \mathbf{2})$*. Let*

$$\mathbb{M} = \langle M; G, R, \mathfrak{T} \rangle ,$$

where

(1) *R is the set of all subuniverses of $\mathbf{M}^2$ which are maximal in*

$$(\omega_1, \omega_2)^{-1}(\leqslant) := \{ (a, b) \in M^2 \mid \omega_1(a) \leqslant \omega_2(b) \},$$

for some $\omega_1, \omega_2 \in \Omega_M$,

(2) *$G \subseteq \mathrm{End}(\mathbf{M})$ satisfies the separation condition:*

(S) for all $a \neq b$ in M, we have $\omega(u(a)) \neq \omega(u(b))$, for some $\omega \in \Omega_M$ and some u in the submonoid of $\mathrm{End}(\mathbf{M})$ generated by G,

(3) *$\mathfrak{T}$ is the discrete topology.*

Then $\mathbb{M}$ yields a duality on $\mathcal{A}$.

We call each element of Ω_M a **carrier** and each element of R a **piggyback relation**. In practice, we try to minimize the size of the set Ω_M at the expense of increasing the size of the set G to reduce the size of R. However, even in the case with only two carriers ω_1 and ω_2, we are not sure that the size of R is small as the number of maximal subuniverses in $(\omega_1, \omega_2)^{-1}(\leqslant)$ could be large. Fortunately, for Ockham algebras, the following lemma shows us that for each pair of carriers ω_1 and ω_2, there is a unique maximal subuniverse in $(\omega_1, \omega_2)^{-1}(\leqslant)$.

Lemma 3.2.2 ([26], Lemma 3.5). *Let* $\mathbf{A} = \langle A; \vee, \wedge, f, 0, 1 \rangle$ *be an Ockham algebra and assume that* B *forms a sublattice of* $\mathbf{A}^\flat$. *Then*

$$B^\circ = \{\, b \in B \mid (\forall k \in \mathbb{N}) f^k(b) \in B \,\}$$

is the largest subuniverse of $\mathbf{A}$ *contained in* B.

We now use these results to obtain a piggyback duality for the quasi-variety generated by a single finite subdirectly irreducible Ockham algebra.

For the rest of this section, fix $m \in \mathbb{N}$ and $n \in \{0, \dots, m-1\}$. Consider the one-generated Ockham space $\mathbb{W} = \langle \{\, 0, \dots, m{-}1 \,\}; \leqslant, g, \mathcal{T} \rangle$ in which $g(i) = i{+}1$, for $i = 0, \dots, m-2$ and $g(m-1) = n$ (see Figure 3.2). Let $\mathbf{M} = K(\mathbb{W})$. Then $\mathbf{M}$ is subdirectly irreducible and every finite subdirectly irreducible Ockham algebra arises in this way (see [62]). It follows that every non-trivial subalgebra of $\mathbf{M}$ is subdirectly irreducible. To find a duality for $\mathsf{ISP}(\mathbf{M})$ using Theorem 3.2.1, we first need to find the sets Ω_M and G that satisfy the separation condition (S) of the Piggyback Duality Theorem 3.2.1.

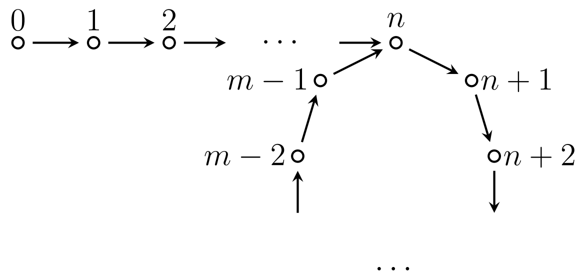


Figure 3.2: The labelling of a one-generated Ockham space

We define $\omega_0, \omega_1 \colon \mathbf{M}^\flat \to \mathbf{2}$ by $\omega_0(\alpha) = \alpha(0)$ and $\omega_1(\alpha) = \alpha(1)$, for all $\alpha \in M = \boldsymbol{\mathcal{P}}(\mathbb{W}, \underline{2})$. Define $u \colon \mathbf{M} \to \mathbf{M}$ by $u(\alpha) = \alpha \circ g^2$. Note that $u = K(g^2)$ is an endomorphism of $\mathbf{M}$ since g^2 is order-preserving (as g is order-reversing).

Lemma 3.2.3. *The sets* $\Omega_M = \{\, \omega_0, \omega_1 \,\}$ *and* $G = \{u\}$ *satisfy the separation condition* (S) *of the Piggyback Duality Theorem.*

Proof. We shall show that if $\alpha \neq \beta$ in $\mathbf{M} = K(\mathbb{W})$ then there exists $j \geqslant 0$ such that $(\omega_0 \circ u^j)(\alpha) \neq (\omega_0 \circ u^j)(\beta)$ or $(\omega_1 \circ u^j)(\alpha) \neq (\omega_1 \circ u^j)(\beta)$. Assume that $\alpha \neq \beta$ in $\mathbf{M}$ and let k be the least element of W such that $\alpha(k) \neq \beta(k)$. We consider two cases.

(1) $k = 2j$, for some $j \geqslant 0$. Then we have

$$\begin{aligned}(\omega_0 \circ u^j)(\alpha) &= \omega_0(u^j(\alpha)) = \omega_0(\alpha \circ g^{2j}) = \omega_0(\alpha \circ g^k) = (\alpha \circ g^k)(0)\\ &= \alpha(g^k(0)) = \alpha(k) \neq \beta(k) = (\omega_0 \circ u^j)(\beta).\end{aligned}$$

(2) $k = 2j+1$, for some $j \geqslant 0$. Then we have

$$\begin{aligned}(\omega_1 \circ u^j)(\alpha) &= \omega_1(u^j(\alpha)) = \omega_1(\alpha \circ g^{2j})\\ &= (\alpha \circ g^{2j})(1) = \alpha(g^{2j}(1)) = \alpha(g^{2j+1}(0))\\ &= \alpha(g^k(0)) = \alpha(k) \neq \beta(k) = (\omega_1 \circ u^j)(\beta).\end{aligned}$$

Hence the sets $\Omega_M = \{\,\omega_0, \omega_1\,\}$ and $G = \{u\}$ satisfy the separation condition (S) of the Piggyback Duality Theorem, as required. □

With two carriers ω_0, ω_1 needed to separate the points of M, it follows from Lemma 3.2.2 that we have four piggyback relations in the structure $\mathbb{M}$, namely $(\omega_i, \omega_j)^{-1}(\leqslant)^\circ$, for $i, j \in \{0,1\}$. We shall now show that these piggyback relations can be obtained from the following alternating order on $\{0,1\}^W$.

Definition 3.2.4. For all $\alpha, \beta \in \{0,1\}^W$, we define

$$\alpha \ll \beta \iff (\forall k \geqslant 0) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & \text{if } k \text{ is even,}\\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & \text{if } k \text{ is odd.}\end{cases}$$

Then $\ll$ is an order on $\{0,1\}^W$ and is known as the **alternating order**. Note that, in the definition of the alternating order, k is a non-negative integer, not just an element of $\mathbb{Z}_m = \{\,0, 1, \ldots, m-1\,\}$. Since $g^m = g^n$, the order $\ll$ changes radically according to whether $m-n$ is even or odd. If $m-n$ is even, then

$$\alpha \ll \beta \iff (\forall k \in \mathbb{Z}_m) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & \text{if } k \text{ is even,} \\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & \text{if } k \text{ is odd,} \end{cases}$$

and if $m-n$ is odd, then

$$\alpha \ll \beta \iff (\forall k \in \mathbb{Z}_m) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & \text{if } k < n \text{ is even,} \\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & \text{if } k < n \text{ is odd,} \\ \alpha(g^k(0)) = \beta(g^k(0)), & \text{if } n \leqslant k \leqslant m-1. \end{cases}$$

The next result uses the order $\ll$ to obtain a piggyback duality for the quasi-variety generated by the subdirectly irreducible Ockham algebra $\mathbf{M} = K(\mathbb{W})$.

Theorem 3.2.5. *Let $\mathbb{W} = \langle \{0, 1, \dots, m-1\}; \leqslant, g, \mathcal{T}\rangle$ be a one-generated Ockham space, generated by 0, and let $\mathbf{M} = K(\mathbb{W})$. Define $u\colon \mathbf{M} \to \mathbf{M}$ by $u(\alpha) = \alpha \circ g^2$, for all $\alpha \in M = \mathcal{P}(\mathbb{W}, 2\!\!2)$. Then*

$$\mathbb{M} = \langle M; u, \ll_0, \ll_1, \ll_{01}, \ll_{10}, \mathcal{T}\rangle,$$

yields a duality on $\mathsf{ISP}(\mathbf{M})$, where

(1) $\ll_0 \subseteq M^2$ *is the restriction of the alternating order on* $\{0,1\}^W$,

(2) $\ll_1 := \{ (\alpha, \beta) \in M^2 \mid \alpha \circ g \ll \beta \circ g \text{ in } \{0,1\}^W \}$,

(3) $\ll_{01} := \{ (\alpha, \beta) \in M^2 \mid \alpha \ll \beta \circ g \text{ in } \{0,1\}^W \}$,

(4) $\ll_{10} := \{ (\alpha, \beta) \in M^2 \mid \alpha \circ g \ll \beta \text{ in } \{0,1\}^W \}$.

Proof. We shall use Theorem 3.2.1, Lemma 3.2.2 and Lemma 3.2.3. Recall that $\omega_0, \omega_1\colon \mathbf{M}^\flat \to \mathbf{2}$ are given by $\omega_0(\alpha) = \alpha(0)$ and $\omega_1(\alpha) = \alpha(1)$. It suffices to prove that the relations $\ll_0, \ll_1, \ll_{01}, \ll_{10}$ are the four piggyback relations, i.e., the largest subuniverses of $(\omega_i, \omega_j)^{-1}(\leqslant)$, for $i, j \in \{0,1\}$. Recall that for

$\alpha \in M = \mathbf{\mathcal{P}}(\mathbb{X}, \underline{2})$, we have $f(\alpha) = c \circ \alpha \circ g$ and hence $f^k(\alpha) = \alpha \circ g^k$, if k is even, and $f^k(\alpha) = c \circ \alpha \circ g^k$, if k is odd. Now, using Lemma 3.2.2 we have

$$
\begin{aligned}
(1) \quad (\alpha, \beta) \in (\omega_0, \omega_0)^{-1}(\leqslant)^\circ &\iff (\forall k \geqslant 0)\ f^k(\alpha, \beta) \in (\omega_0, \omega_0)^{-1}(\leqslant) \\
&\iff (\forall k \geqslant 0)\ \omega_0(f^k(\alpha)) \leqslant \omega_0(f^k(\beta)) \\
&\iff (\forall k \geqslant 0)\ f^k(\alpha)(0) \leqslant f^k(\beta)(0) \\
&\iff (\forall k \geqslant 0) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & k \text{ is even}, \\ c(\alpha(g^k(0))) \leqslant c(\beta(g^k(0))), & k \text{ is odd}, \end{cases} \\
&\iff (\forall k \geqslant 0) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & k \text{ is even}, \\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & k \text{ is odd}, \end{cases} \\
&\iff \alpha \ll \beta \text{ in } \{0,1\}^W.
\end{aligned}
$$

Thus $(\omega_0, \omega_0)^{-1}(\leqslant)^\circ = \ll_0$.

$$
\begin{aligned}
(2) \quad (\alpha, \beta) \in (\omega_1, \omega_1)^{-1}(\leqslant)^\circ &\iff (\forall k \geqslant 0)\ f^k(\alpha, \beta) \in (\omega_1, \omega_1)^{-1}(\leqslant) \\
&\iff (\forall k \geqslant 0)\ \omega_1(f^k(\alpha)) \leqslant \omega_1(f^k(\beta)) \\
&\iff (\forall k \geqslant 0)\ f^k(\alpha)(1) \leqslant f^k(\beta)(1) \\
&\iff (\forall k \geqslant 0) \begin{cases} \alpha(g^k(1)) \leqslant \beta(g^k(1)), & k \text{ even}, \\ c(\alpha(g^k(1))) \leqslant c(\beta(g^k(1))), & k \text{ odd}, \end{cases} \\
&\iff (\forall k \geqslant 0) \begin{cases} \alpha(g^k(1)) \leqslant \beta(g^k(1)), & k \text{ even}, \\ \alpha(g^k(1)) \geqslant \beta(g^k(1)), & k \text{ odd}, \end{cases} \\
&\iff (\forall k \geqslant 0) \begin{cases} (\alpha \circ g)(g^k(0)) \leqslant (\beta \circ g)(g^k(0)), & k \text{ even}, \\ (\alpha \circ g)(g^k(0)) \geqslant (\beta \circ g)(g^k(0)), & k \text{ odd}, \end{cases} \\
&\iff \alpha \circ g \ll \beta \circ g \text{ in } \{0,1\}^W.
\end{aligned}
$$

Thus $(\omega_1, \omega_1)^{-1}(\leqslant)^\circ = \ll_1$.

$$
\begin{aligned}
(3)\quad (\alpha,\beta)\in(\omega_0,\omega_1)^{-1}(\leqslant)^\circ &\iff (\forall k\geqslant 0)\ f^k(\alpha,\beta)\in(\omega_0,\omega_1)^{-1}(\leqslant)\\
&\iff (\forall k\geqslant 0)\ \omega_0(f^k(\alpha))\leqslant\omega_1(f^k(\beta))\\
&\iff (\forall k\geqslant 0)\ f^k(\alpha)(0)\leqslant f^k(\beta)(1)\\
&\iff (\forall k\geqslant 0)\begin{cases}\alpha(g^k(0))\leqslant\beta(g^k(1)), & k \text{ even},\\ c(\alpha(g^k(0)))\leqslant c(\beta(g^k(1))), & k \text{ odd},\end{cases}\\
&\iff (\forall k\geqslant 0)\begin{cases}\alpha(g^k(0))\leqslant\beta(g^k(1)), & k \text{ even},\\ \alpha(g^k(0))\geqslant\beta(g^k(1)), & k \text{ odd},\end{cases}\\
&\iff (\forall k\geqslant 0)\begin{cases}\alpha(g^k(0))\leqslant(\beta\circ g)(g^k(0)), & k \text{ even},\\ \alpha(g^k(0))\geqslant(\beta\circ g)(g^k(0)), & k \text{ odd},\end{cases}\\
&\iff \alpha\ll\beta\circ g \text{ in } \{0,1\}^W.
\end{aligned}
$$

Thus $(\omega_0,\omega_1)^{-1}(\leqslant)^\circ = \ll_{01}$.

$$
\begin{aligned}
(4)\quad (\alpha,\beta)\in(\omega_1,\omega_0)^{-1}(\leqslant)^\circ &\iff (\forall k\geqslant 0)\ f^k(\alpha,\beta)\in(\omega_1,\omega_0)^{-1}(\leqslant)\\
&\iff (\forall k\geqslant 0)\ \omega_1(f^k(\alpha))\leqslant\omega_0(f^k(\beta))\\
&\iff (\forall k\geqslant 0)\ f^k(\alpha)(1)\leqslant f^k(\beta)(0)\\
&\iff (\forall k\geqslant 0)\begin{cases}\alpha(g^k(1))\leqslant\beta(g^k(0)), & k \text{ even},\\ c(\alpha(g^k(1)))\leqslant c(\beta(g^k(0))), & k \text{ odd},\end{cases}\\
&\iff (\forall k\geqslant 0)\begin{cases}\alpha(g^k(1))\leqslant\beta(g^k(0)), & k \text{ even},\\ \alpha(g^k(1))\geqslant\beta(g^k(0)), & k \text{ odd},\end{cases}\\
&\iff (\forall k\geqslant 0)\begin{cases}(\alpha\circ g)(g^k(0))\leqslant\beta(g^k(0)), & k \text{ even},\\ (\alpha\circ g)(g^k(0))\geqslant\beta(g^k(0)), & k \text{ odd},\end{cases}\\
&\iff \alpha\circ g\ll\beta \text{ in } \{0,1\}^W.
\end{aligned}
$$

Thus $(\omega_1,\omega_0)^{-1}(\leqslant)^\circ = \ll_{10}$. □

Example 3.2.6. We consider the one-generated Ockham space $\mathbb{W}$ given in Figure 3.3. Then $\mathbf{K} := K(\mathbb{W})$ is the 3-element Kleene algebra. The unusual order on W, with $0 > 1$, is chosen so that the alternating order on K will agree with the familiar uncertainty order. (The uncertainty order was introduced by Mukaidono [49]; see also Mukaidono [50] and Brzozowski and Negulescu [11].)

We shall write an element $\alpha \in K = \mathcal{P}(\mathbb{W}, \mathbb{2})$ as the string $\alpha(0)\alpha(1)$ with $\alpha(0), \alpha(1) \in \{0, 1\}$. We have the following table.

$\alpha \in K$	$f(\alpha) = c \circ \alpha \circ g$	$u(\alpha) = \alpha \circ g^2$	$\alpha \circ g$
$0 = 00$	$11 = 1$	$00 = 0$	00
$a = 10$	$10 = a$	$10 = a$	01
$1 = 11$	$00 = 0$	$11 = 1$	11

In this case, $n = 0$ and $m = 2$ (see Figure 3.2). Hence $m - n$ is even and by Definition 3.2.4, the alternating order $\ll$ is given on $\{0, 1\}^W$ by, for all $\alpha, \beta \in \{0, 1\}^W$,

$$\alpha \ll \beta \iff (\forall k \in \mathbb{Z}_m) \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & \text{if } k \text{ is even,} \\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & \text{if } k \text{ is odd.} \end{cases}$$

Thus $\alpha \ll \beta \iff \alpha(0) \leqslant \beta(0)$ and $\alpha(1) \geqslant \beta(1)$. We work out the four piggyback relations using Theorem 3.2.5 and the table above as follows:

$$\begin{aligned}
\ll_0 &= \{ (\alpha, \beta) \in K^2 \mid \alpha \ll \beta \text{ in } \{0,1\}^W \} \\
&= \{ (0,0), (0,a), (a,a), (1,a), (1,1) \}, \\
\ll_1 &= \{ (\alpha, \beta) \in K^2 \mid \alpha \circ g \ll \beta \circ g \text{ in } \{0,1\}^W \} \\
&= \{ (0,0), (a,0), (a,a), (a,1), (1,1) \} = \text{ the converse of } \ll_0, \\
\ll_{01} &= \{ (\alpha, \beta) \in K^2 \mid \alpha \ll \beta \circ g \text{ in } \{0,1\}^W \} \\
&= \{ (0,0), (1,1) \} = \triangle_{K_0}, \text{where } K_0 = \{0,1\}, \\
\ll_{10} &= \{ (\alpha, \beta) \in K^2 \mid \alpha \circ g \ll \beta \text{ in } \{0,1\}^W \} \\
&= \{(0,0), (0,a), (a,0), (a,a), (a,1), (1,a), (1,1)\} = K^2 \setminus \{(0,1), (1,0)\} =: \sim.
\end{aligned}$$

Observe from the above table that u is the identity on $\mathbf{K}$ so we do not need to include it in the alter ego. Thus, by Theorem 3.2.5, $\mathbb{K} = \langle \{0, a, 1\}; \ll_0, K_0, \sim, \mathcal{T} \rangle$ yields a duality on $\mathsf{ISP}(\mathbf{K})$. This is the duality obtained originally by Davey and Werner [32] by studying all subalgebras of $\mathbf{K}^2$. The approach used here was given in Davey and Priestley [26].

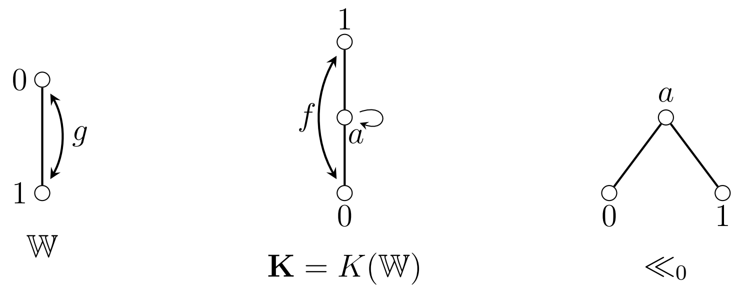


Figure 3.3: The 3-element Kleene algebra $\mathbf{K}$ and the alternating order

We now show that in the case g is order-preserving, we need only one carrier, that is ω_0, and one endomorphism to meet the separation condition (S) from Theorem 3.2.1.

Lemma 3.2.7. *Let* $\mathbb{W} = \langle \{0, 1, \dots, m-1\}; \leqslant, g, \mathcal{T} \rangle$ *be a one-generated Ockham space, generated by* 0*, and let* $\mathbf{M} = K(\mathbb{W})$*. Assume that* g *is order-preserving. Define* $d\colon M \to M$ *by* $d(\alpha) = \alpha \circ g$*, for all* $\alpha \in M = \mathcal{P}(\mathbb{W}, 2)$*, and* $\omega_0\colon \mathbf{M}^\flat \to \mathbf{2}$ *by* $\omega_0(\alpha) = \alpha(0)$*, for all* $\alpha \in M$*. Then the sets* $\Omega_M = \{\omega_0\}$ *and* $G = \{d\}$ *satisfy the separation condition* (S)*.*

Proof. As g is an endomorphism of $\mathbb{W}$, the map $d = K(g)$ is an endomorphism of $\mathbf{M}$. Now assume that $\alpha, \beta \in M$ with $\alpha \neq \beta$ and let $k \in \{0, 1, \dots, m-1\}$ such that $\alpha(k) \neq \beta(k)$. Then we have

$$(\omega_0 \circ d^k)(\alpha) = \omega_0(d^k(\alpha)) = \omega_0(\alpha \circ g^k) = \alpha(g^k(0)) = \alpha(k) \neq \beta(k) = (\omega_0 \circ d^k)(\beta).$$

Hence the sets Ω_M and G satisfy the separation condition (S), as required. $\square$

Using Theorem 3.2.1, Lemma 3.2.2 and Lemma 3.2.7 we shall obtain a natural duality via an alter ego with one unary map and one binary relation. In fact, we shall prove that this structure yields a strong duality on the quasi-variety generated by $\mathbf{M}$.

Theorem 3.2.8. *Let* $\mathbb{W} = \langle \{0, 1, \ldots, m-1\}; \leqslant, g, \mathcal{T}\rangle$ *be a one-generated Ockham space, generated by* 0*, in which* g *is order-preserving. Let* $\mathbf{M} = K(\mathbb{W})$ *and define* $d\colon \mathbf{M} \to \mathbf{M}$ *by* $d(\alpha) = \alpha \circ g$*, for all* $\alpha \in M = \mathcal{P}(\mathbb{W}, 2)$*. Then*

$$\mathbb{M} = \langle M; d, \ll, \mathcal{T}\rangle$$

yields a strong duality for $\mathsf{ISP}(\mathbf{M})$*, where* $\ll\, \subseteq M^2$ *is the restriction of the alternating order on* $\{0,1\}^W$*.*

Proof. Observe that by Theorem 3.2.1, Lemma 3.2.2, Lemma 3.2.7 and calculation (1) in the proof of Theorem 3.2.5, $\mathbb{M} = \langle M; d, \ll, \mathcal{T}\rangle$ already yields a duality on $\mathsf{ISP}(\mathbf{M})$. Recall that every non-trivial subalgebra of $\mathbf{M}$ is subdirectly irreducible. In addition, we have $\mathbf{M}$ is injective in $\boldsymbol{\mathcal{A}} := \mathsf{ISP}(\mathbf{M})$, by Goldberg [36]; see also [20]. It then follows from Corollary 1.3.8 that $\mathbb{M}'_{\mathcal{T}} = \langle M; \{d\} \cup \mathrm{End}(\mathbf{M}), \ll, \mathcal{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{A}}$. Thus to prove $\mathbb{M}$ yields a strong duality on $\mathsf{ISP}(\mathbf{M})$, it suffices to show that d generates $\mathrm{End}(\mathbf{M})$, i.e.,

$$\mathrm{End}(\mathbf{M}) = \{d^k \mid k \geqslant 0\}.$$

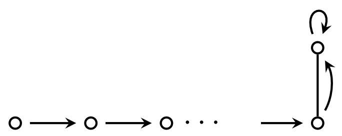
Note that as g is order-preserving, we have $g^k \in \mathrm{End}(\mathbb{W})$, for all $k \geqslant 0$. Now let $e \in \mathrm{End}(\mathbf{M}) = \mathrm{End}(K(\mathbb{W}))$. Then $e = K(\hat{e})$ for some $\hat{e} \in \mathrm{End}(\mathbb{W})$. Define $k := \hat{e}(0)$. Then $k \in \{0, \ldots, m-1\}$. We have $\hat{e}(0) = k = g^k(0)$ and hence $\hat{e} = g^k$, for some $k \in \{0, \ldots, m-1\}$, as $\mathbb{W}$ is generated by 0 as an Ockham space. It follows that $e = K(\hat{e}) = K(g^k) = d^k$. Hence d generates $\mathrm{End}(\mathbf{M})$, as required. □

The next result gives the finite one-generated Ockham spaces with g order-preserving.

Lemma 3.2.9 ([26, 3.9])**.** *Let* $\mathbb{W} = \langle \{0, 1, \dots, m-1\}; \leqslant, g, \mathcal{T} \rangle$ *be a one-generated Ockham space, generated by* 0*. The map* g *is order-preserving in the following cases:*

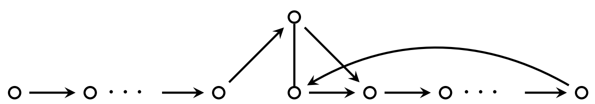
(1) $\mathbb{W}$ *is an antichain;*

(2) $\mathbb{W}$ *is isomorphic to*



or its order-theoretic dual;

(3) $\mathbb{W}$ *is isomorphic to*



or its order-theoretic dual.

3.3 An example

In this section, we shall investigate a family of one-generated Ockham spaces, given in Figure 3.4, that plays an important role in the next chapter, namely $\mathbb{D}_m$, for each m odd. Observe that $\mathbb{D}_m$ is a special case of the Ockham space given in Lemma 3.2.9(3). As g is order-preserving, we can apply Theorem 3.2.8 to find an alter ego that yields a strong duality on the quasi-variety generated by $K(\mathbb{D}_m)$.

Example 3.3.1. Fix an odd number m. Let $\mathbf{S}_m$ denote the Ockham algebra with dual space $\mathbb{D}_m$, i.e., $\mathbf{S}_m = K(\mathbb{D}_m)$. It then follows from Theorem 3.2.8 that the structure $\mathbb{S}_m = \langle S_m; d, \ll, \mathcal{T} \rangle$ yields a strong duality on $\mathsf{ISP}(\mathbf{S}_m)$. Explicit descriptions of $\ll$ and d are given in (3.1) and (3.2) below.

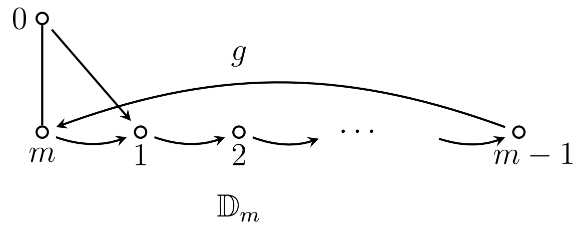


Figure 3.4: The Ockham space $\mathbb{D}_m$

For each $\alpha \in S_m$, α is an order-preserving map from $\mathbb{D}_m$ to $\mathbb{2}$. We denote each such α by the string $\alpha(0)\alpha(1)\dots\alpha(m)$, where $\alpha(i) \in \{0,1\}$, for each $i \in \{0,1,\dots,m\}$. Since the cycle $\mathbb{D}_m$ has size m, which is odd, the alternating order $\ll$ on S_m is given by:

$$\alpha \ll \beta \iff \begin{cases} \alpha(g^k(0)) \leqslant \beta(g^k(0)), & \text{if } k < n \text{ is even,} \\ \alpha(g^k(0)) \geqslant \beta(g^k(0)), & \text{if } k < n \text{ is odd,} \\ \alpha(g^k(0)) = \beta(g^k(0)), & \text{if } n \leqslant k \leqslant m, \end{cases}$$

for all $\alpha, \beta \in S_m = \mathcal{P}(\mathbb{D}_m, \mathbb{2})$. With $n = 1$, that means

$$\alpha \ll \beta \iff \alpha(0) \leqslant \beta(0) \text{ and } \alpha(i) = \beta(i), \text{ for all } i \in \{1,\dots,m\}. \tag{3.1}$$

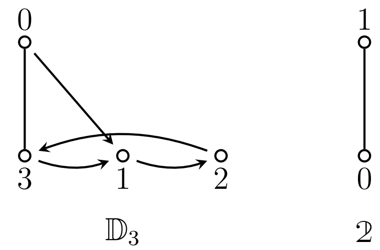
It follows from (3.1) that, as an ordered set, $\mathbb{S}_m$ is a disjoint union of one-element and two-element chains. The endomorphism d is given by $d(\alpha) = \alpha \circ g$, for each $\alpha \in S_m$. Thus

$$d(\alpha(0)\alpha(1)\dots\alpha(m)) = \alpha((g(0))\dots\alpha(g(m)) = \alpha(1)\alpha(2)\dots\alpha(m)\alpha(1). \tag{3.2}$$

As $\mathbb{S}_m$ yields a strong duality on $\mathsf{ISP}(\mathbf{S}_m)$, it follows that (IC) holds. Hence, by Lemma 1.2.10, every compatible relation on $\mathbf{S}_m$ is equivalent to one of the form $\mathrm{E}(\mathbb{X})\restriction_S$, where $\mathbb{X} \in \mathsf{ISP}_\mathsf{f}(\mathbb{S}_m)$ and S is a non-empty generating set for $\mathbb{X}$. We shall give a concrete description for $\mathsf{ISP}_\mathsf{f}(\mathbb{S}_m)$ in the next chapter (Theorem 4.3.2), where it is used to represent compatible relations on $\mathbf{S}_m$.

Remark 3.3.2. In the case $m = 1$, the algebra $\mathbf{S}_1$ is the 3-element dual Stone algebra and the alter ego $\mathbb{S}_1$ yields the usual natural duality for the variety of dual Stone algebras [16, 17]; see also [13, 4.3.6].

In the case $m = 3$, we can work out the algebra $\mathbf{S}_3 = K(\mathbb{D}_3)$ and the alter ego $\mathbb{S}_3 = \langle S_3; d, \ll, \mathcal{T}\rangle$ of $\mathbf{S}_3$ by using the following table. Recall that each element α of S_3 is an order-preserving map from $\mathbb{D}_3$ to $\mathbb{2}$. Once again, we write an element $\alpha \in S_3 = \mathcal{P}(\mathbb{D}_3, \mathbb{2})$ as the string $\alpha(0)\alpha(1)\alpha(2)\alpha(3)$ with $\alpha(i) \in \{0, 1\}$, for all $i \in \{0, 1, 2, 3\}$.



$\alpha \in S_3$	$d(\alpha) = \alpha \circ g$	$f(\alpha) = c \circ \alpha \circ g$
$0 = 0000$	$0000 = 0$	$1111 = 1$
$\alpha_1 = 0100$	$1001 = \alpha_7$	$0110 = \alpha_4$
$\alpha_2 = 0010$	$0100 = \alpha_1$	$1011 = \alpha_{10}$
$\alpha_3 = 1000$	$0000 = 0$	$1111 = 1$
$\alpha_4 = 0110$	$1101 = \alpha_9$	$0010 = \alpha_2$
$\alpha_5 = 1100$	$1001 = \alpha_7$	$0110 = \alpha_4$
$\alpha_6 = 1010$	$0100 = \alpha_1$	$1011 = \alpha_{10}$
$\alpha_7 = 1001$	$0010 = \alpha_2$	$1101 = \alpha_9$
$\alpha_8 = 1110$	$1101 = \alpha_9$	$0010 = \alpha_2$
$\alpha_9 = 1101$	$1011 = \alpha_{10}$	$0100 = \alpha_1$
$\alpha_{10} = 1011$	$0110 = \alpha_4$	$1001 = \alpha_7$
$1 = 1111$	$1111 = 1$	$0000 = 0$

Using the above table and the restricted Priestley duality, the picture for $\mathbf{S}_3 = \langle S_3; \vee, \wedge, f, 0, 1\rangle$ is given in Figure 3.5. The arrows represent the unary operation f. Recall that $f(0) = 1$ and $f(1) = 0$.

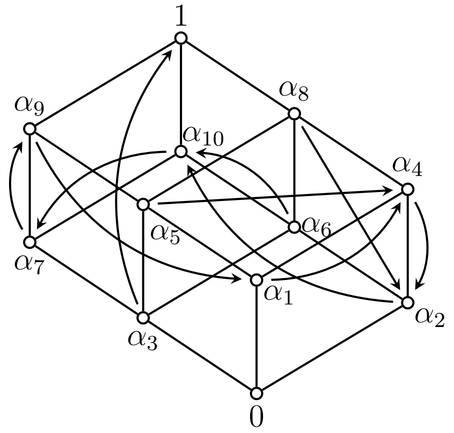


Figure 3.5: The algebra $\mathbf{S}_3 = K(\mathbb{D}_3)$

Using the above table and Equation (3.1), we have

$$\ll = \triangle_{S_3} \cup \{(0, \alpha_3), (\alpha_1, \alpha_5), (\alpha_2, \alpha_6), (\alpha_4, \alpha_8)\}.$$

and hence the diagram for the alter ego $\mathbb{S}_3 = \langle S_3; d, \ll, \mathcal{T}\rangle$ is given in Figure 3.6.

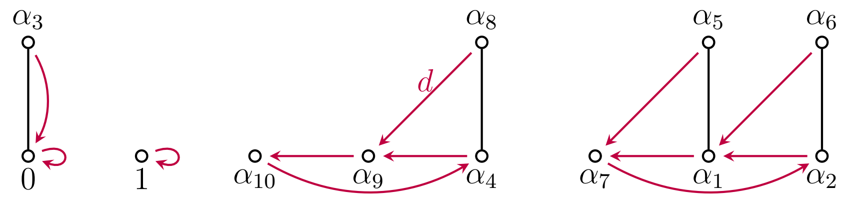


Figure 3.6: The alter ego $\mathbb{S}_3$ of $\mathbf{S}_3$

Chapter 4

Compatible relations on Ockham algebras

In this chapter, we shall complete the characterisation within the variety $\mathcal{O}$ of Ockham algebras. We find all of the finite Ockham algebras that admit only finitely many compatible relations; up to isomorphism, there are countably infinitely many and all are subdirectly irreducible. Our characterisation is stated in terms of Ockham dual spaces (Theorem 4.1.1) in the first section. The following sections will be the proof of the theorem. The characterisations for the familiar subvarieties $\mathcal{S}, \mathcal{K}, \mathcal{M}$ and $\mathcal{MS}$, which are given in the final section, are easily obtained from this result.

4.1 The characterisation for Ockham algebras

In this section, we state the characterisation theorem and present some lemmas that will be needed in the proof. Recall that, for structures $\mathbb{X}, \mathbb{Y}$ of the same type, if $\mathbb{X} \in \mathsf{HS}(\mathbb{Y})$, then $\mathbb{X}$ is a divisor of $\mathbb{Y}$.

Theorem 4.1.1. *Let* **A** *be a non-trivial finite Ockham algebra. Then the following are equivalent:*

(1) **A** *admits only finitely many relations;*

(2) *there is an odd number* m *such that the dual space* $H(\mathbf{A})$ *is isomorphic to* $\mathbb{C}_m$, $\mathbb{D}_m$ *or* $\mathbb{D}_m^\partial$ *from Figure 4.1;*

(3) *the dual space* $H(\mathbf{A})$ *does not have as a divisor any of the eight Ockham spaces from Figure 4.2.*

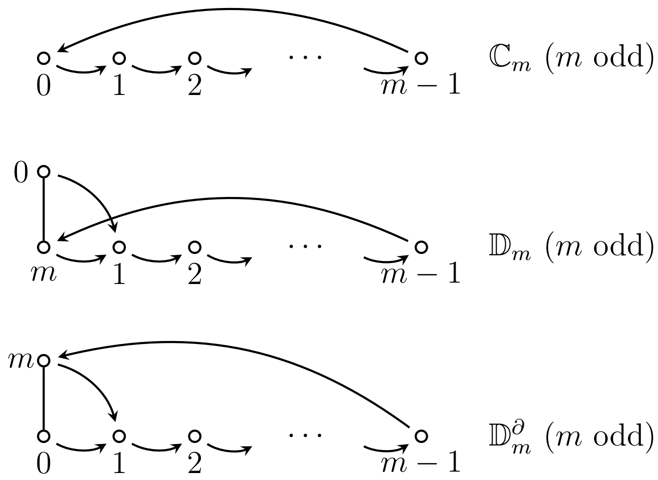


Figure 4.1: Duals of the non-trivial Ockham algebras with only finitely many relations

Up to symmetry, the Ockham algebras with only finitely many relations can be grouped into two infinite families.

The first family generalises the 2-element Boolean algebra (which has dual space $\mathbb{C}_1$). We shall show that the Ockham algebras with dual spaces $\mathbb{C}_1$, $\mathbb{C}_3$, $\mathbb{C}_5$, … are precisely the quasi-primal Ockham algebras (Theorem 4.2.6). It is known that every quasi-primal algebra admits only finitely many relations [25, 2.10].

The second family generalises the 3-element Stone algebra (which has dual space $\mathbb{D}_1^\partial$). Recall from Chapter 3 that $\mathbf{S}_m$ is the Ockham algebra with dual space $\mathbb{D}_m$, for each odd number m. We shall use the piggyback duality given in Example 3.3.1 to encode the compatible relations on $\mathbf{S}_m$ and thereby show that $\mathbf{S}_m$ admits only finitely many relations.

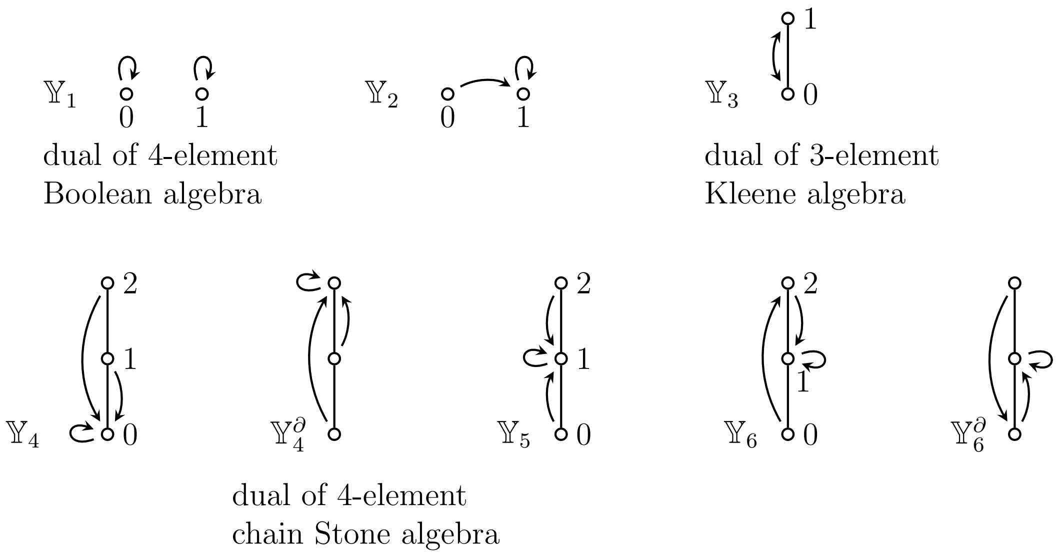


Figure 4.2: The eight dual space obstacles

To prove the main theorem, we first need the following lemma. It is the natural restriction of the corresponding result from Priestley duality for the variety $\boldsymbol{\mathcal{O}}$ (see [13, 7.4.1]).

Lemma 4.1.2. *A homomorphism* $\varphi\colon \mathbf{A} \to \mathbf{B}$ *in* $\boldsymbol{\mathcal{O}}$ *is an embedding (respectively, a surjection) if and only if its dual morphism* $H(\varphi)\colon H(\mathbf{B}) \to H(\mathbf{A})$ *in* $\boldsymbol{\mathcal{Y}}$ *is a surjection (respectively, an embedding).*

This lemma leads easily to the following result, which will allow us to apply the Transfer Lemma 1.2.14 to Ockham algebras from within the dual class $\boldsymbol{\mathcal{Y}}$ of Ockham spaces.

Lemma 4.1.3. *Let* $\mathbf{A}, \mathbf{B} \in \boldsymbol{\mathcal{O}}$*. Then* $\mathbf{A}$ *is a divisor of* $\mathbf{B}$ *if and only if* $H(\mathbf{A})$ *is a divisor of* $H(\mathbf{B})$*.*

Proof. Assume $\mathbf{A} \in \mathsf{HS}(\mathbf{B})$. Then there is an embedding $\varphi\colon \mathbf{C} \hookrightarrow \mathbf{B}$ and a surjection $\psi\colon \mathbf{C} \twoheadrightarrow \mathbf{A}$. It follows from Lemma 4.1.2 that there is a surjection $H(\varphi)\colon H(\mathbf{B}) \twoheadrightarrow H(\mathbf{C})$ and an embedding $H(\psi)\colon H(\mathbf{A}) \hookrightarrow H(\mathbf{C})$. So $H(\mathbf{A}) \in \mathsf{SH}(H(\mathbf{B})) \subseteq \mathsf{HS}(H(\mathbf{B}))$.

Now assume $H(\mathbf{A}) \in \mathsf{HS}(H(\mathbf{B}))$. Since the categories $\mathfrak{O}$ and $\mathfrak{Y}$ are dually equivalent, there is some $\mathbf{C} \in \mathfrak{O}$ with an embedding $H(\varphi)\colon H(\mathbf{C}) \hookrightarrow H(\mathbf{B})$ and a surjection $H(\psi)\colon H(\mathbf{C}) \twoheadrightarrow H(\mathbf{A})$. Using Lemma 4.1.2 again, we have $\varphi\colon \mathbf{B} \twoheadrightarrow \mathbf{C}$ and $\psi\colon \mathbf{A} \hookrightarrow \mathbf{C}$. So $\mathbf{A} \in \mathsf{SH}(\mathbf{B}) \subseteq \mathsf{HS}(\mathbf{B})$. □

The following lemma, which is used in the next section, generalises part of Lemma 4.1.2 (cf. [13, 7.4.1]).

Lemma 4.1.4. *A homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}_1 \times \mathbf{B}_2$ in $\mathfrak{O}$ is an embedding if and only if the two morphisms $H(\pi_i \circ \varphi)\colon H(\mathbf{B}_i) \to H(\mathbf{A})$ in $\mathfrak{Y}$, for $i \in \{1, 2\}$, are jointly surjective.*

Proof. For $i \in \{1, 2\}$, let $\pi_i\colon \mathbf{B}_1 \times \mathbf{B}_2 \to \mathbf{B}_i$ denote the ith projection. Then $H(\pi_1) \dot\cup H(\pi_2)\colon H(\mathbf{B}_1) \dot\cup H(\mathbf{B}_2) \to H(\mathbf{B}_1 \times \mathbf{B}_2)$ is an isomorphism, as coproduct in $\mathfrak{Y}$ is given by disjoint union. Define the homomorphism $\varphi_i := \pi_i \circ \varphi\colon \mathbf{A} \to \mathbf{B}_i$. Then $H(\varphi_i) = H(\pi_i \circ \varphi) = H(\varphi) \circ H(\pi_i)$. So the following diagram commutes.

$$\begin{array}{ccc} H(\mathbf{B}_1 \times \mathbf{B}_2) & \xrightarrow{H(\varphi)} & H(\mathbf{A}) \\ {\scriptstyle H(\pi_1) \dot\cup H(\pi_2)} \updownarrow & \nearrow {\scriptstyle H(\varphi_1) \dot\cup H(\varphi_2)} & \\ H(\mathbf{B}_1) \dot\cup H(\mathbf{B}_2) & & \end{array}$$

So $H(\varphi)$ is surjective if and only if $H(\varphi_1)$ and $H(\varphi_2)$ are jointly surjective. The claim follows because $H(\varphi)$ is surjective if and only if φ is an embedding, by Lemma 4.1.2. □

4.2 Quasi-primal Ockham algebras

In this section, we show that an Ockham algebra is quasi-primal if and only if its dual space is isomorphic to $\mathbb{C}_m$ from Figure 4.1, for some odd m.

We say that an algebra $\mathbf{M}$ is **quasi-primal** if $\mathbf{M}$ generates an arithmetical variety and every non-trivial subalgebra of $\mathbf{M}$ is simple. A **partial automorphism** of an algebra is an isomorphism between two of its subalgebras. In this section, we use the following characterisation of quasi-primality. This result is a part of the result from [13, Theorem 3.3.12].

Theorem 4.2.1. *For a finite algebra* $\mathbf{M}$*, the following are equivalent:*

(1) $\mathbf{M}$ *is quasi-primal;*

(2) $\mathbf{M}$ *has a ternary near-unanimity term and every subalgebra of* $\mathbf{M}^2$ *is either a product of two subalgebras of* $\mathbf{M}$ *or the graph of a partial automorphism of* $\mathbf{M}$*.*

We also use the following description of the binary compatible relations on an Ockham algebra. This lemma has been used in the literature (see [26]) but not proved. So we include a proof for completeness.

Lemma 4.2.2. *Let* $\mathbb{X}$ *be an Ockham space and let* $\mathbf{r} \leqslant K(\mathbb{X})^2$*. Then*

$$r = \{\, (\alpha \circ \varphi_1, \alpha \circ \varphi_2) \mid \alpha \in KH(\mathbf{r}) \,\},$$

for some jointly surjective morphisms $\varphi_1, \varphi_2 \colon \mathbb{X} \to H(\mathbf{r})$*.*

Proof. Define $\mathbf{M} := K(\mathbb{X})$. Then $\mathbf{r} \leqslant \mathbf{M}^2$. Let $\rho_1, \rho_2 \colon \mathbf{r} \to \mathbf{M}$ denote the two projections. The inclusion $\rho_1 \sqcap \rho_2 \colon \mathbf{r} \to \mathbf{M}^2$ is an embedding, and therefore the morphisms $H(\rho_1), H(\rho_2) \colon H(\mathbf{M}) \to H(\mathbf{r})$ are jointly surjective, by Lemma 4.1.4.

As $\mathbf{M} = K(\mathbb{X})$, we define the morphism $\varphi_i \colon \mathbb{X} \to H(\mathbf{r})$ by $\varphi_i := H(\rho_i) \circ \varepsilon_{\mathbb{X}}$, for each $i \in \{1, 2\}$. Since $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to H(\mathbf{M})$ is an isomorphism, the morphisms $\varphi_1, \varphi_2 \colon \mathbb{X} \to H(\mathbf{r})$ are jointly surjective.

Since $e_{\mathbf{r}} \colon \mathbf{r} \to KH(\mathbf{r})$ is an isomorphism, we have

$$\begin{aligned} r &= \{\, \big(\rho_1(a), \rho_2(a)\big) \mid a \in r \,\} \\ &= \{\, \big(\rho_1 \circ e_{\mathbf{r}}^{-1}(\alpha),\ \rho_2 \circ e_{\mathbf{r}}^{-1}(\alpha)\big) \mid \alpha \in KH(\mathbf{r}) \,\}. \end{aligned}$$

So it remains to check that $\rho_i \circ e_{\mathbf{r}}^{-1}(\alpha) = \alpha \circ \varphi_i$, for $i \in \{1, 2\}$ and for all $\alpha \in KH(\mathbf{r})$. Since $\langle H, K, e, \varepsilon \rangle$ is a dual adjunction between $\mathbfcal{O}$ and $\mathbfcal{Y}$, we have

$\rho_i = K(H(\rho_i) \circ \varepsilon_{\mathbb{X}}) \circ e_{\mathbf{r}}$; see [13, Figure 1.2]. Thus

$$\rho_i \circ e_{\mathbf{r}}^{-1}(\alpha) = K(H(\rho_i) \circ \varepsilon_{\mathbb{X}})(\alpha) = \alpha \circ H(\rho_i) \circ \varepsilon_{\mathbb{X}} = \alpha \circ \varphi_i,$$

as required. □

We next prove some basic facts about cycles in Ockham spaces that will be used in this section and in the final section.

Definition 4.2.3. Let $\mathbb{X} = \langle X; \leqslant, g, \mathfrak{T}\rangle$ be an Ockham space and let $C \subseteq X$. For $m \in \mathbb{N}$, we will say that C is an m**-cycle** of $\mathbb{X}$ if we can enumerate C as $c_0, \dots, c_{m-1}$ such that $g(c_i) = c_{i+1 \pmod m}$. In this case, we say that C is an **odd cycle** if m is odd, and an **even cycle** otherwise. Note that a 1-cycle of $\mathbb{X}$ is just a fixed point of g.

Lemma 4.2.4. *Let $\mathbb{X}$ be an Ockham space such that every element belongs to an odd cycle. Then $\mathbb{X}$ is an antichain.*

Proof. Let $c, d \in X$ with c in an m-cycle and d in an n-cycle, for some odd m and n. Assume that $c \leqslant d$. As m and n are odd and g is order-reversing, we have

$$g^m(d) \leqslant g^m(c) = c \quad \text{and} \quad d = g^n(d) \leqslant g^n(c).$$

As $m + n$ is even, it now follows that

$$d \leqslant g^n(c) = g^{m+n}(c) \leqslant g^{m+n}(d) = g^m(d) \leqslant c.$$

Thus $c = d$. Hence $\mathbb{X}$ is an antichain. □

Lemma 4.2.5. *Let $\mathbb{X}$ be an Ockham space that contains an even cycle. Then the Ockham space $\mathbb{Y}_3$ from Figure 4.2 is a divisor of $\mathbb{X}$.*

Proof. Assume $\mathbb{C} \leqslant \mathbb{X}$ such that C is an m-cycle, for some even m. Since C is finite and $g{\restriction}_C : C \to C$ is an order-reversing bijection, it follows that $g{\restriction}_C$ is a dual order-automorphism of $\mathbb{C}$. So g sends maximal elements of C to minimal elements of C, and vice versa. Let x be a maximal element of C. Then we have $C = \{\, g^k(x) \mid k \in \{0, 1, \dots, m-1\} \,\}$. As m is even, we can define $\varphi\colon \mathbb{C} \twoheadrightarrow \mathbb{Y}_3$ by

$$\varphi(g^k(x)) = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

So $\mathbb{Y}_3 \in \mathsf{HS}(\mathbb{X})$, as required. □

Theorem 4.2.6. *An Ockham algebra is quasi-primal if and only if its dual space is isomorphic to $\mathbb{C}_m$ from Figure 4.1, for some odd m.*

Proof. Let m be odd. We first show that the Ockham algebra $\mathbf{M} := K(\mathbb{C}_m)$ is quasi-primal. As $\mathbf{M}$ is lattice-based and therefore has a ternary near-unanimity term, it suffices to show that every subalgebra of $\mathbf{M}^2$ is the product of two subalgebras of $\mathbf{M}$ or the graph of a partial automorphism of $\mathbf{M}$ (by Theorem 4.2.1).

Let $\mathbf{r} \leqslant \mathbf{M}^2$. Then, using Lemma 4.2.2, there are jointly surjective morphisms $\varphi_1, \varphi_2\colon \mathbb{C}_m \to H(\mathbf{r})$ such that

$$r = \{\, (\alpha \circ \varphi_1, \alpha \circ \varphi_2) \mid \alpha \in KH(\mathbf{r}) \,\}.$$

Since C_m is an odd cycle and $\varphi_1, \varphi_2\colon \mathbb{C}_m \to H(\mathbf{r})$ are jointly surjective, it follows that the Ockham space $H(\mathbf{r})$ is either an odd cycle or the union of two different odd cycles. We consider these two cases separately.

Case 1: *$H(\mathbf{r})$ is an odd cycle.*

In this case, the morphism $\varphi_i\colon \mathbb{C}_m \to H(\mathbf{r})$ is surjective, for each $i \in \{1, 2\}$. We shall show that r is the graph of a partial automorphism of $\mathbf{M}$. Let $a, b, c \in M$ with $(a, b), (a, c) \in r$. Then there exist $\alpha, \beta \in KH(\mathbf{r})$ such that $a = \alpha \circ \varphi_1$, $b = \alpha \circ \varphi_2$ and $a = \beta \circ \varphi_1$, $c = \beta \circ \varphi_2$. So $\alpha \circ \varphi_1 = a = \beta \circ \varphi_1$. Since φ_1 is surjective, we must have $\alpha = \beta$ and hence $b = c$. By symmetry, if $(a, b), (c, b) \in r$, then $a = c$.

Case 2: *$H(\mathbf{r})$ is the union of two different odd cycles.*

By Lemma 4.2.4, the Ockham space $H(\mathbf{r})$ is an antichain. So we can write

$H(\mathbf{r}) = \mathbb{X}_1 \dot\cup \mathbb{X}_2$, where $\mathbb{X}_i$ is an odd cycle with $\varphi_i \colon \mathbb{C}_m \to \mathbb{X}_i$. It follows that

$$\begin{aligned} r &= \{\, (\alpha \circ \varphi_1, \alpha \circ \varphi_2) \mid \alpha \in KH(\mathbf{r}) \,\} \\ &= \{\, \alpha_1 \circ \varphi_1 \mid \alpha_1 \in K(\mathbb{X}_1) \,\} \times \{\, \alpha_2 \circ \varphi_2 \mid \alpha_2 \in K(\mathbb{X}_2) \,\}. \end{aligned}$$

So $\mathbf{r}$ is the product of two subalgebras of $\mathbf{M}$.

Now assume that $\mathbf{M}$ is a quasi-primal Ockham algebra. Then $\mathbf{M}$ is simple. By Lemma 4.1.2, this implies that $H(\mathbf{M})$ has no non-empty proper substructures. So $H(\mathbf{M})$ must be an m-cycle, for some $m \in \mathbb{N}$. Suppose that m is even. Then $\mathbb{Y}_3$ is a divisor of $H(\mathbf{M})$, by Lemma 4.2.5. So the 3-element Kleene algebra $\mathbf{K} = K(\mathbb{Y}_3)$ belongs to the variety generated by $\mathbf{M}$, by Lemma 4.1.3. But $\mathbf{K}$ generates the variety of all Kleene algebras, which is not congruence permutable. This contradicts our assumption that $\mathbf{M}$ is quasi-primal. Therefore m is odd. So $H(\mathbf{M})$ is isomorphic to $\mathbb{C}_m$, by Lemma 4.2.4. □

Every quasi-primal algebra admits only finitely many relations [25, 2.10]. So the Ockham algebra $\mathbf{M} = K(\mathbb{C}_m)$ admits only finitely many relations, for each odd m. We obtain an alternative proof of this in the next section.

4.3 Ockham algebras with only finitely many relations

In this section, we prove the implication (2) ⇒ (1) in Theorem 4.1.1. Note that $\mathbb{C}_m$ is a divisor of $\mathbb{D}_m$, for each odd m. So, using symmetry and Lemmas 1.2.14 and 4.1.3, it suffices to show that the Ockham algebra $\mathbf{S}_m := K(\mathbb{D}_m)$ admits only finitely many relations, for each odd m.

Recall from Example 3.3.1 that the structure $\mathbb{S}_m := \langle \boldsymbol{\mathcal{P}}(\mathbb{D}_m, \mathbb{2}); d, \ll, \mathcal{T} \rangle$ strongly dualises $\mathbf{S}_m$ and therefore satisfies (IC), where

- d is the endomorphism of $\mathbf{S}_m$ given by $d(\alpha) := \alpha \circ g$, for all $\alpha \in S_m$,

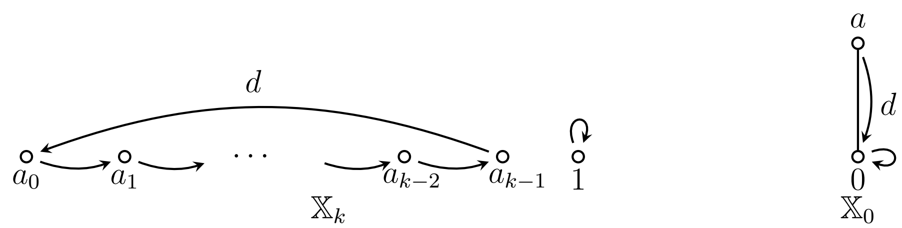
- $\ll$ is the order on $S_m = \boldsymbol{\mathcal{P}}(\mathbb{D}_m, \mathbb{2})$ given by $\alpha \ll \beta$ if and only if $\alpha(0) \leqslant \beta(0)$ and $\alpha{\restriction}_{D_m\setminus\{0\}} = \beta{\restriction}_{D_m\setminus\{0\}}$, and
- $\mathcal{T}$ is the discrete topology.

It then follows from Lemma 1.2.10 that every compatible relation on $\mathbf{S}_m$ is equivalent to one of the form $\mathrm{E}(\mathbb{X}){\restriction}_S := \{\, \alpha{\restriction}_S \mid \alpha\colon \mathbb{X} \to \mathbb{S}_m \,\} \subseteq S_m^S$, where $\mathbb{X} \in \mathsf{ISP_f}(\mathbb{S}_m)$ and S is a non-empty generating set for $\mathbb{X}$. To be able to make use of this description of the compatible relations on $\mathbf{S}_m$, we next develop a concrete description of the structures in $\mathsf{ISP_f}(\mathbb{S}_m)$. In fact, we shall describe the topological structures in the dual class $\mathsf{IS}_\mathsf{c}^0\mathsf{P}^+(\mathbb{S}_m)$. We first come to the following result.

Lemma 4.3.1. *Let m be odd and let $\mathbb{S}_m = \langle S_m; d, \ll, \mathcal{T}\rangle$ be the alter ego of $\mathbf{S}_m$ from Example 3.3.1.*

(1) *For each divisor k of m, the structure $\mathbb{X}_k = \langle \{a_0, \dots, a_{k-1}\} \cup \{1\}; d, \ll, \mathcal{T}\rangle$ shown below embeds into $\mathbb{S}_m$.*

(2) *The structure $\mathbb{X}_0 = \langle \{0, a\}; d, \ll, \mathcal{T}\rangle$ shown below embeds into $\mathbb{S}_m$.*

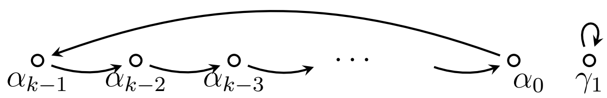


Proof. (1) Let k be a divisor of m. We shall prove that $\mathbb{X}_k$ embeds into $\mathbb{S}_m$. Recall that S_m consists of all order-preserving maps from $\mathbb{D}_m$ to $\mathbb{2}$. We first prove that $\mathbb{X}_1$ embeds into $\mathbb{S}_m$. Define the order-preserving maps γ_0 and γ_1 from $\mathbb{D}_m$ to $\mathbb{2}$ by $\gamma_0(i) = 0$ and $\gamma_1(i) = 1$, for $i \in \{0, 1, \dots, m\}$. Then it follows easily from equations (3.1) and (3.2) that $\gamma_0 \parallel \gamma_1$, $d(\gamma_0) = \gamma_0$, and $d(\gamma_1) = \gamma_1$. So $\langle \{\gamma_0, \gamma_1\}; d, \ll, \mathcal{T}\rangle \in \mathsf{S}(\mathbb{S}_m)$ and hence $\mathbb{X}_1 \in \mathsf{IS}(\mathbb{S}_m)$.

Now we assume that $k \geqslant 3$. For each $\ell \in \{0, 1, \ldots, k-1\}$, define $\alpha_\ell \colon D_m \to 2$ by

$$\alpha_\ell(i) = \begin{cases} 1, & \text{if } i \equiv \ell \pmod{k}, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\alpha_\ell(0) = \alpha_\ell(m)$, for each $\ell \in \{0, 1, \ldots, k-1\}$, as $m \equiv 0 \pmod{k}$. It follows that α_ℓ is an order-preserving map for all $\ell \in \{0, 1, \ldots, k-1\}$. Now define the order-preserving map $\gamma_1 \colon \mathbb{D}_m \to \mathbb{2}$ by $\gamma_1(i) = 1$, for all $i \in D_m$. We want to prove that the subset $Y := \{\alpha_0, \ldots, \alpha_{k-1}\} \cup \{\gamma_1\}$ of S_m forms a substructure $\mathbb{Y}$ of $\mathbb{S}_m$ and that $\mathbb{Y}$ has the following structure.



Let $j, \ell \in \{0, 1, \ldots, k-1\}$ with $j \neq \ell$. Then $\alpha_j(j) = 1 \neq 0 = \alpha_\ell(j)$ and so $\alpha_j \not\leqslant \alpha_\ell$. Define $I := \{1, \ldots, k-1\} \setminus \{j\}$. Then we have $I \neq \varnothing$, as $k \geqslant 3$. Note that $I \subseteq D_m \setminus \{0\}$. For $i \in I$, we have $\alpha_j(i) = 0 \neq 1 = \gamma_1(i)$. So $\alpha_j \parallel \gamma_1$, by equation (3.1). Therefore Y is an antichain.

Next we will show that $d(\alpha_j) = \alpha_\ell$, where $\ell \equiv j-1 \pmod{k}$, and $d(\gamma_1) = \gamma_1$. Clearly, $d(\gamma_1) = \gamma_1$, by equation (3.2). Let $j, \ell \in \{0, 1, \ldots, k-1\}$ with $\ell \equiv j-1 \pmod{k}$. Then using equation (3.2) again gives, for $i \in \{0, 1, \ldots, m-1\}$,

$$\begin{aligned} d(\alpha_j)(i) = \alpha_j(i+1) &= \begin{cases} 1, & \text{if } i+1 \equiv j \pmod{k}, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & \text{if } i \equiv j-1 \pmod{k}, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & \text{if } i \equiv \ell \pmod{k}, \text{ (as } \ell \equiv j-1 \pmod{k}) \\ 0, & \text{otherwise,} \end{cases} \\ &= \alpha_\ell(i), \end{aligned}$$

and

$$
\begin{aligned}
d(\alpha_j)(m) = \alpha_j(1) &= \begin{cases} 1, & \text{if } 1 \equiv j \pmod{k}, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1, & \text{if } 0 \equiv j-1 \pmod{k}, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1, & \text{if } m \equiv \ell \pmod{k}, \text{ (as } \ell \equiv j-1 \text{ and } m \equiv 0 \pmod{k}) \\ 0, & \text{otherwise,} \end{cases} \\
&= \alpha_\ell(m).
\end{aligned}
$$

Thus, $d(\alpha_j) = \alpha_\ell$, where $\ell \equiv j - 1 \pmod{k}$.

It now follows that $\mathbb{Y} = \langle \{\alpha_0, \ldots, \alpha_{k-1}\} \cup \{\gamma_1\}; d, \ll, \mathcal{T} \rangle \in \mathsf{S}(\mathbb{S}_m)$ and hence $\mathbb{X}_k \in \mathsf{IS}(\mathbb{S}_m)$.

(2) Define the order-preserving maps γ_0 and γ_a from $\mathbb{D}_m$ to $\mathbb{2}$ by $\gamma_0(i) = 0$, for all $i \in \{0, 1, \ldots, m\}$, and $\gamma_a(0) = 1, \gamma_a(i) = 0$, for $i \neq 0$. Using equations (3.1) and (3.2), it is easy to check that $\gamma_0 \ll \gamma_a$ and $d(\gamma_0) = d(\gamma_a) = \gamma_0$. It follows that $\langle \{\gamma_0, \gamma_a\}; d, \ll, \mathcal{T} \rangle \in \mathsf{S}(\mathbb{S}_m)$ and hence $\mathbb{X}_0 \in \mathsf{IS}(\mathbb{S}_m)$. □

We now give a description of the structures in $\mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{S}_m)$.

Theorem 4.3.2. *Let m be odd and let $\mathbb{X} = \langle X; d, \ll, \mathcal{T} \rangle$ be a topological structure of the same type as $\mathbb{S}_m$. Then $\mathbb{X} \in \mathsf{IS}^0_\mathsf{c}\mathsf{P}^+(\mathbb{S}_m)$ if and only if the following hold.*

(1) *$\langle X; \ll, \mathcal{T} \rangle$ is a Priestley space,*

(2) *$\mathbb{X}$ satisfies $x \ll y \Rightarrow d(x) \approx d(y)$, and*

(3) *$\mathbb{X}$ satisfies $d^m(x) \ll x$.*

Moreover, it follows from (1)–(3) that

(4) *if x is minimal, then $d^m(x) = x$,*

(5) *each $\ll$-connected component of $\mathbb{X}$ has a least element, and*

(6) *d sends each element of $\mathbb{X}$ to a minimal element.*

Proof. Assume that $\mathbb{X} \in \mathsf{IS}_\mathsf{c}^0\mathsf{P}^+(\mathbb{S}_m)$. Clearly, $\langle X; \ll, \mathfrak{T}\rangle$ is a Priestley space, as $\ll$ is an order on S_m. Now we check that $\mathbb{S}_m$ satisfies the two quasi-equations (2) and (3).

Let $\alpha, \beta \in S_m = \boldsymbol{\mathcal{P}}(\mathbb{D}_m, \mathbb{2})$ with $\alpha \ll \beta$. Then $\alpha(0) \leqslant \beta(0)$ and $\alpha(i) = \beta(i)$ for all $i \in D_m \setminus \{0\}$, by equation (3.1). For all $i \in D_m$, we have $g(i) \neq 0$ and so $d(\alpha)(i) = \alpha(g(i)) = \beta(g(i)) = d(\beta)(i)$. Thus $d(\alpha) = d(\beta)$. Hence $\mathbb{S}_m$ satisfies the quasi-equation (2).

Let $\alpha \in S_m$. Then

$$\begin{aligned} d^m(\alpha) = d^m(\alpha(0)\alpha(1)\dots\alpha(m)) &= \alpha(g^m(0))\alpha(g^m(1))\dots\alpha(g^m(m)) \\ &= \alpha(m)\alpha(1)\alpha(2)\dots\alpha(m) \\ &\ll \alpha(0)\alpha(1)\alpha(2)\dots\alpha(m) = \alpha \end{aligned}$$

as $\alpha\colon \mathbb{D}_m \to \mathbb{2}$ is order-preserving. Thus $d^m(\alpha) \ll \alpha$ for all $\alpha \in S_m$. Thus $\mathbb{S}_m$ satisfies (3). Since $\mathbb{X} \in \mathsf{IS}_\mathsf{c}^0\mathsf{P}^+(\mathbb{S}_m)$, the structure $\mathbb{X}$ also satisfies (2) and (3), by the Preservation Theorem [13, 1.4.3].

Conversely, assume that $\mathbb{X}$ satisfies (1)–(3). We first show that $\mathbb{X}$ satisfies (4)–(6). Condition (4) follows directly from (1) and (3). Next, note that, as $\mathbb{X}$ is a Priestley space, every element of X is greater than or equal to a minimal element, by Remark 1.3.2(1). Let x and y be in the same $\ll$-connected component such that x and y are minimal elements. Then $d(x) = d(y)$, by (2). So $d^m(x) = d^m(y)$. Since x and y are minimal, $d^m(x) = x$ and $d^m(y) = y$, by (4). Thus $x = y$. Hence each $\ll$-connected component of $\mathbb{X}$ has a unique minimal element and therefore (5) holds.

Now let $x \in X$ and assume that there is $y \in X$ such that $y \ll d(x)$. Then (2) and (3) together give $d^{m+1}(x) = d(x)$. It then follows from (2) and (3) that $y \gg d^m(y) = d^{m+1}(x) = d(x)$. Thus $y = d(x)$ and so $d(x)$ is minimal and hence (6) holds, as required.

We now prove that $\mathbb{X} \in \mathsf{IS}_\mathsf{c}^0\mathsf{P}^+(\mathbb{S}_m)$ using the Separation Theorem 1.3.9.

Since $x \neq y$ if and only if $x \not\ll y$ or $y \not\ll x$, it suffices to find morphisms separating $\ll$.

Let $\mathrm{Min}(\mathbb{X})$ be the set of minimal elements of $\mathbb{X}$. Clearly, $\mathrm{Min}(\mathbb{X})$ is an antichain and is a down-set in X. Note by (4) and (6) that $\mathrm{Min}(\mathbb{X}) = d(X)$. And also by (4), for all $z \in \mathrm{Min}(\mathbb{X})$, we have $d^m(z) = z$. So $d{\restriction_{\mathrm{Min}(\mathbb{X})}} \colon \mathrm{Min}(\mathbb{X}) \to \mathrm{Min}(\mathbb{X})$ is a bijection. Since d is continuous and X is compact Hausdorff, it follows (see Lemma [13, B.1]) that the map d is closed. Hence $\mathrm{Min}(\mathbb{X}) = d(X)$ is a closed subset of X and $d{\restriction_{\mathrm{Min}(\mathbb{X})}} \colon \mathrm{Min}(\mathbb{X}) \to \mathrm{Min}(\mathbb{X})$ is a homeomorphism.

Now let $x, y \in X$ with $x \not\ll y$. We shall use the substructures $\mathbb{X}_k$ and $\mathbb{X}_0$ of $\mathbb{S}_m$ given in Lemma 4.3.1 to separate the alternating order $\ll$. We consider the following two cases.

Case (1): $x \notin \mathrm{Min}(\mathbb{X})$.

The subsets ${\uparrow}x$ and ${\downarrow}y \cup \mathrm{Min}(\mathbb{X})$ are closed in X, by Remark 1.3.2(2). Since ${\uparrow}x \cap ({\downarrow}y \cup \mathrm{Min}(\mathbb{X})) = \varnothing$, it then follows from Remark 1.3.2(3) that there exists a clopen up-set U in X such that $x \in U, y \notin U$ and $\mathrm{Min}(\mathbb{X}) \cap U = \varnothing$. Define the map $\varphi \colon X \to X_0$ by

$$\varphi(z) := \begin{cases} a, & \text{if } z \in U, \\ 0, & \text{if } z \in X \setminus U. \end{cases}$$

Then it is easy to check that φ is a morphism from $\mathbb{X}$ to $\mathbb{X}_0$ and we have $\varphi(x) = a \not\ll 0 = \varphi(y)$.

Case (2): $x \in \mathrm{Min}(\mathbb{X})$.

We first show that $d(x) \neq d(y)$. Suppose that $d(x) = d(y)$. Then since $x \in \mathrm{Min}(\mathbb{X})$, we have $x = d^m(x) = d^m(y) \ll y$, by (3) and (4). This is a contradiction, as $x \not\ll y$ by the assumption. Therefore $d(x) \neq d(y)$. We can define $x_1 = d(x)$ and $y_1 = d(y)$ in $d(X) = \mathrm{Min}(\mathbb{X})$; say that x_1 is of order k, i.e., $(\forall i \in \mathbb{N})\ d^i(x_1) = x_1 \iff k \mid i$. Since $d^m(x_1) = x_1$, by (4), we have $k \mid m$.

Let U be a clopen subset of $\mathrm{Min}(\mathbb{X})$ with $x_1 \in U$ and with

$$U \cap \{y_1, d(x_1), \dots, d^{k-1}(x_1)\} = \varnothing.$$

Since $d{\restriction}_{\mathrm{Min}(\mathbb{X})}\colon \mathrm{Min}(\mathbb{X}) \to \mathrm{Min}(\mathbb{X})$ is a homeomorphism, we can define a clopen subset V of U by

$$\begin{aligned} V := U \cap \bigcap \big\{d^i(U) \mid i \in \{1, \dots, m\} \text{ with } k | \, i\big\} \\ \setminus \bigcup \big\{d^i(U) \mid i \in \{1, \dots, m\} \text{ with } k \not| i\big\}. \end{aligned}$$

From the definition of V, it is straightforward to see that $x_1 \in V$ and $y_1 \notin V$.

Claim (★): $(\forall v \in V)\ (\forall j \in \mathbb{N})\ d^j(v) \in V \iff k | \, j$.

Since $d^m(v) = v$, for all $v \in V$, it suffices to show that

$$(\forall v \in V)\ (\forall j \in \{1, \dots, m\})\ d^j(v) \in V \iff k | \, j.$$

Let $v \in V$ and let $j \in \{1, \dots, m\}$. Note that, for all $i \in \mathbb{N}$, we have $v \in d^i(U) \iff k | \, i$, as $d^m(U) = U$ and $k | \, m$. Now let $i \in \{1, \dots, m\}$. Then

$$\begin{aligned} d^j(v) \in d^i(U) &\iff d^j(v) \in d^{i+m}(U) \\ &\iff v \in d^{i+m-j}(U) \qquad (\blacklozenge) \\ &\iff k | \, i + m - j \\ &\iff k | \, i - j. \end{aligned}$$

We now prove Claim (★). First, assume that $d^j(v) \in V$. As $v \in U$, we have $d^j(v) \in V \cap d^j(U)$. So $k | \, j$, by the definition of V.

Conversely, assume that $k | \, j$. We need to prove that $d^j(v) \in V$, i.e., to prove:

(i) $d^j(v) \in U$;

(ii) $d^j(v) \in d^i(U)$, for all $i \in \{1, \dots, m\}$ with $k | \, i$; and

(iii) $d^j(v) \notin d^i(U)$, for all $i \in \{1, \dots, m\}$ with $k \not| i$.

Observe that, (i) is just a special case of (ii) in which $i = m$. We now prove (ii). Let $i \in \{1, \dots, m\}$ with $k|\, i$. Then we have $k|\, i + m - j$. It then follows from (♦) that $d^j(v) \in d^i(U)$, as required. To prove (iii), let $i \in \{1, \dots, m\}$ with $k \not|\, i$. Then $k \not|\, i - j$, as $k|\, j$. Using (♦) again, we have $d^j(v) \notin d^i(U)$.

It now follows from Claim (★) that $d^k(V) \subseteq V$ and $d^i(V) \cap d^j(V) = \varnothing$, for $i, j \in \{1, \dots, k-1\}$ with $i \neq j$. With these facts and that the map $d{\restriction}_{\mathrm{Min}(\mathbb{X})}\colon \mathrm{Min}(\mathbb{X}) \to \mathrm{Min}(\mathbb{X})$ is a bijection, the map $\varphi\colon \mathrm{Min}(\mathbb{X}) \to X_k$ given by

$$\varphi(z) := \begin{cases} a_0, & \text{if } z \in V, \\ a_i, & \text{if } z \in d^i(V), \text{ for some } i \in \{1, \dots, k-1\}, \\ 1, & \text{if } z \in \mathrm{Min}(\mathbb{X}) \setminus \big(V \cup \bigcup \{ d^i(V) \mid i \in \{1, \dots, k-1\}\}\big) \end{cases}$$

is a well-defined morphism from $\mathrm{Min}(\mathbb{X})$ to $\mathbb{X}_k$. Note that $d\colon \mathbb{X} \to \mathrm{Min}(\mathbb{X})$ is a morphism, by condition (2). Then the morphism $\varphi \circ d\colon \mathbb{X} \to \mathbb{X}_k$ separates $x \not\ll y$, since we have that X_k is an antichain, $(\varphi \circ d)(x) = \varphi(x_1) = a_0$ and $(\varphi \circ d)(y) = \varphi(y_1) \neq a_0$, as $y_1 \notin V$.

Thus $\mathbb{X} \in \mathsf{IS}^0_{\mathsf{c}}\mathsf{P}^+(\mathbb{S}_m)$, by the Separation Theorem 1.3.9. □

Remark 4.3.3. We shall say that a structure $\mathbb{X}$ in $\mathsf{ISP_f}(\mathbb{S}_m)$ is **d-connected** if, for all $x, y \in X$, there exist $m, n \in \mathbb{N}$ such that $d^m(x) = d^n(y)$. It follows from 4.3.2(2) that each finite structure in $\mathsf{ISP_f}(\mathbb{S}_m)$ can be written as a disjoint union of d-connected substructures.

Now consider a d-connected structure $\mathbb{X}$ in $\mathsf{ISP_f}(\mathbb{S}_m)$. We shall show that $\mathbb{X}$ must have the general shape shown in Figure 4.3. Let m_0 be a $\ll$-minimal element of $\mathbb{X}$. It follows from 4.3.2(4) that $m_0 = d^m(m_0)$. Choose $k \in \mathbb{N}$ as small as possible such that $m_0 = d^k(m_0)$. Then k is a divisor of m. For each $i \in \{1, \dots, k-1\}$, define $m_i := d^i(m_0)$ and let P_i be the up-set of $\mathbb{X}$ generated by m_i. Since $\mathbb{X}$ is d-connected, it follows from 4.3.2(4), (6) that $m_0, \dots, m_{k-1}$ are precisely the minimal elements of $\mathbb{X}$. Using 4.3.2(2), it follows that $d(P_i) = \{m_{i+1 \ (\mathrm{mod}\ k)}\}$, for all $i \in \{0, \dots, k-1\}$. So the $\ll$-connected components of $\mathbb{X}$ are $P_0, \ldots, P_{k-1}$, and therefore $\mathbb{X}$ has the shape shown in Figure 4.3.

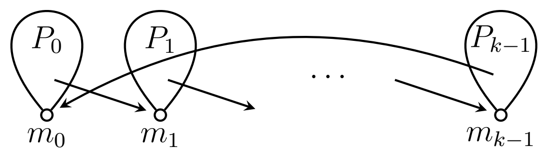
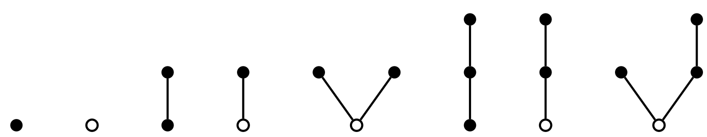


Figure 4.3: The shape of a d-connected structure

The next lemma restricts the number (up to equivalence) of compatible relations on $\mathbf{S}_m$ that come from d-connected structures in $\mathsf{ISP_f}(\mathbb{S}_m)$.

Lemma 4.3.4. *Let m be odd, and let $\mathbb{X}$ be a d-connected structure in $\mathsf{ISP_f}(\mathbb{S}_m)$ with generating set S. Then $\mathrm{E}(\mathbb{X}){\restriction_S}$ is equivalent to the relation $\mathrm{E}(\mathbb{Y}){\restriction_{S\cap Y}}$, for some $\mathbb{Y} \leqslant \mathbb{X}$ such that each $\ll$-connected component of $\mathbb{Y}$ has one of the following eight forms (with elements of S shaded):*



Proof. Assume $\mathbb{X}$ has a $\ll$-connected component that does not have one of the eight allowable forms shown above. We will prove that there is a proper substructure $\mathbb{Y}$ of $\mathbb{X}$ such that the two relations $\mathrm{E}(\mathbb{X}){\restriction_S}$ and $\mathrm{E}(\mathbb{Y}){\restriction_{S\cap Y}}$ are equivalent, where $S \cap Y$ is a generating set for $\mathbb{Y}$. Since $\mathbb{X}$ is finite, the result will follow by induction.

By Remark 4.3.3, the structure $\mathbb{X} = \langle X; d, \ll\rangle$ has the shape shown in Figure 4.3, for some divisor k of m. Define $\mathrm{Min}(\mathbb{X}) := \{m_0, \ldots, m_{k-1}\}$. Then $X \backslash \mathrm{Min}(\mathbb{X}) \subseteq S$, as $d(X) = \mathrm{Min}(\mathbb{X})$ and the set S generates $\mathbb{X}$. In particular, we have $P_0 \backslash \{m_0\} \subseteq S$. Let $\mathbb{P}_0 = \langle P_0; \ll\rangle$ be the induced ordered set on the $\ll$-component P_0 of $\mathbb{X}$. Without loss of generality, we can assume that $\mathbb{P}_0$ does

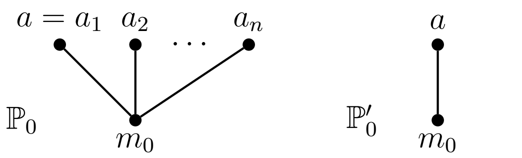
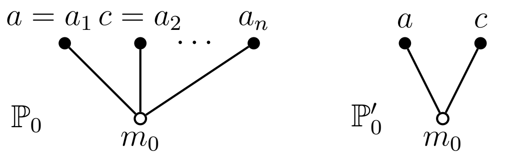
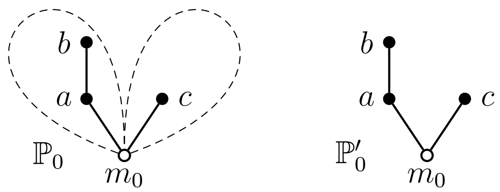
Case 1: $\mathbb{P}_0$ has height 1, $m_0 \in S$ and $\lvert P_0\rvert \geqslant 3$.	
Case 2: $\mathbb{P}_0$ has height 1, $m_0 \notin S$ and $\lvert P_0\rvert \geqslant 4$.	
Case 3: $\mathbb{P}_0$ has height at least 2 and $m_0 \in S$.	
Case 4: $\mathbb{P}_0$ has height at least 2, $m_0 \notin S$, and $P_0 \backslash \{m_0\}$ is order-connected.	
Case 5: $\mathbb{P}_0$ has height at least 2, $m_0 \notin S$, and $P_0 \backslash \{m_0\}$ is order-disconnected.	

Table 4.1: Choosing the subset P_0' of P_0

not have one of the eight allowable forms. So one of the five cases described in Table 4.1 must apply.

Depending on which case applies, choose a subset $\{a\}$, $\{a, b\}$, $\{a, c\}$ or $\{a, b, c\}$ of P_0 according to the appropriate diagram of $\mathbb{P}_0$ in Table 4.1. The sub-ordered set $\mathbb{P}_0'$ of $\mathbb{P}_0$ shown in Table 4.1 is properly contained in $\mathbb{P}_0$, as we are assuming that $\mathbb{P}_0$ does not have one of the eight allowable forms.

Now define the proper substructure $\mathbb{Y}$ of $\mathbb{X}$ by $Y := (X\backslash P_0) \cup P_0'$ and define $T := Y \cap S$. Then T is a generating set for $\mathbb{Y}$. We shall prove that the relations $\mathrm{E}(\mathbb{X})\restriction_S$ and $\mathrm{E}(\mathbb{Y})\restriction_T$ on $\mathbf{S}_m$ are equivalent.

Claim 1: $\mathrm{E}(\mathbb{Y})\restriction_T$ *is conjunct-atomic definable from* $\mathrm{E}(\mathbb{X})\restriction_S$.

We use Lemma 1.2.13. In each of the five cases in Table 4.1, it is easy to find an order-preserving map $\rho_0 \colon \mathbb{P}_0 \to \mathbb{P}'_0$ such that $\rho_0{\restriction}_{P'_0} = \mathrm{id}_{P'_0}$ and $\rho_0^{-1}(m_0) = \{m_0\}$. We can then define $\rho \colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ by $\rho = \rho_0 \cup \mathrm{id}_{X \backslash P_0}$, with $\rho{\restriction}_Y = \mathrm{id}_Y$ and $\rho(S) \subseteq T$. Hence $\mathrm{E}(\mathbb{Y}){\restriction}_T$ is conjunct-atomic definable from $\mathrm{E}(\mathbb{X}){\restriction}_S$, by Lemma 1.2.13.

Claim 2: $\mathrm{E}(\mathbb{X}){\restriction}_S$ *is conjunct-atomic definable from* $\mathrm{E}(\mathbb{Y}){\restriction}_T$.

We shall use Lemma 1.2.12. Let $\varphi \colon S \to S_m$ such that φ does not extend to a morphism from $\mathbb{X}$ to $\mathbb{S}_m$. We begin by checking that one of the following four conditions holds:

(a) $\varphi{\restriction}_T$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$;

(b) there are $w, x \in P_0 \backslash \{m_0\}$ with $w \ll x$ but $\varphi(w) \not\ll \varphi(x)$;

(c) $m_0 \in S$ and there is $x \in P_0 \backslash \{m_0\}$ with $\varphi(m_0) \not\ll \varphi(x)$;

(d) $m_0 \notin S$ and there are $w, x \in P_0 \backslash \{m_0\}$ such that $\varphi(w)$ and $\varphi(x)$ belong to different $\ll$-components of $\mathbb{S}_m$.

To see that one of these conditions holds, assume that (a) fails. Then $\varphi{\restriction}_T$ extends to a morphism $\psi \colon \mathbb{Y} \to \mathbb{S}_m$. Note that $X \backslash S \subseteq \mathrm{Min}(\mathbb{X}) \subseteq Y$. So $X = Y \cup S$. Since $Y \cap S = T$, we can define the map $\chi \colon X \to S_m$ by $\chi = \psi \cup \varphi$. Since χ extends φ, it cannot be a morphism from $\mathbb{X}$ to $\mathbb{S}_m$. Since $\mathrm{Min}(\mathbb{X}) \subseteq Y$ and $\psi \colon \mathbb{Y} \to \mathbb{S}_m$ is a morphism, it follows that $\chi{\restriction}_{P_0}$ is not $\ll$-preserving. We have $m_0, a \in P'_0 \subseteq Y$ and we know that $\psi(m_0)$ is a minimal element of $\mathbb{S}_m$ with $\psi(m_0) \ll \psi(a)$. Since $\chi{\restriction}_{P_0}$ is not $\ll$-preserving, it follows that (b), (c) or (d) must hold.

We have shown that one of the conditions (a)–(d) holds. In each of these four cases, we will find a morphism $\omega \colon \mathbb{Y} \to \mathbb{X}$ with $\omega(T) \subseteq S$ such that the map $\varphi \circ \omega{\restriction}_T \colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{M}$. It will then follow by Lemma 1.2.12 that $\mathrm{E}(\mathbb{X}){\restriction}_S$ is conjunct-atomic definable from $\mathrm{E}(\mathbb{Y}){\restriction}_T$, as required.

Case (a): *$\varphi{\restriction}_T$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$.*

Let $\omega\colon \mathbb{Y} \to \mathbb{X}$ be the inclusion. Then $\omega(T) = T \subseteq S$ and the map $\varphi\circ\omega{\restriction}_T = \varphi{\restriction}_T$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$.

Case (b): *there are $w, x \in P_0\backslash\{m_0\}$ with $w \ll x$ but $\varphi(w) \not\ll \varphi(x)$.*

Only Cases 3, 4 and 5 from Table 4.1 can apply. So we can define the morphism $\omega\colon \mathbb{Y} \to \mathbb{X}$ by

$$\omega(y) = \begin{cases} x, & \text{if } y = b, \\ w, & \text{if } y = a, \\ y, & \text{otherwise.} \end{cases}$$

Then $\omega(T) \subseteq S$. The map $\varphi \circ \omega{\restriction}_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$, because $a \ll b$ in $\mathbb{Y}$ but $\varphi \circ \omega(a) = \varphi(w) \not\ll \varphi(x) = \varphi \circ \omega(b)$ in $\mathbb{S}_m$.

Case (c): *$m_0 \in S$ and there is $x \in P_0\backslash\{m_0\}$ with $\varphi(m_0) \not\ll \varphi(x)$.*

Only Cases 1 and 3 from Table 4.1 can apply. Define the morphism $\omega_1\colon \mathbb{Y} \to \mathbb{X}$ in Case 1 and the morphism $\omega_3\colon \mathbb{Y} \to \mathbb{X}$ in Case 3 by

$$\omega_1(y) = \begin{cases} x, & \text{if } y = a, \\ y, & \text{otherwise,} \end{cases} \quad \text{and} \quad \omega_3(y) = \begin{cases} x, & \text{if } y \in \{a, b\}, \\ y, & \text{otherwise.} \end{cases}$$

Then $\omega_i(T) \subseteq S$. The map $\varphi \circ \omega_i{\restriction}_T\colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$, because $m_0 \ll a$ in $\mathbb{Y}$ but $\varphi\circ\omega_i(m_0) = \varphi(m_0) \not\ll \varphi(x) = \varphi\circ\omega_i(a)$ in $\mathbb{S}_m$.

Case (d): *$m_0 \notin S$ and there are $w, x \in P_0\backslash\{m_0\}$ such that $\varphi(w), \varphi(x)$ belong to different $\ll$-components of $\mathbb{S}_m$.*

By Case (b), we can assume that φ is $\ll$-preserving on $P_0\backslash\{m_0\}$. So only Cases 2 and 5 from Table 4.1 can apply. Define $\omega_2\colon \mathbb{Y} \to \mathbb{X}$ in Case 2 and $\omega_5\colon \mathbb{Y} \to \mathbb{X}$ in Case 5 by

$$\omega_2(y) = \begin{cases} w, & \text{if } y = a, \\ x, & \text{if } y = c, \\ y, & \text{otherwise,} \end{cases} \quad \text{and} \quad \omega_5(y) = \begin{cases} w, & \text{if } y \in \{a, b\}, \\ x, & \text{if } y = c, \\ y, & \text{otherwise.} \end{cases}$$

Then $\varphi \circ \omega_i{\restriction}_T \colon T \to M$ does not extend to a morphism from $\mathbb{Y}$ to $\mathbb{S}_m$, because a and c belong to the same $\ll$-component P_0' of $\mathbb{Y}$, but $w = \varphi \circ \omega_i(a)$ and $x = \varphi \circ \omega_i(c)$ belong to different $\ll$-components of $\mathbb{S}_m$. $\square$

Lemma 4.3.5. *For m odd, the Ockham algebra $\mathbf{S}_m$ dual to $\mathbb{D}_m$ admits only finitely many relations.*

Proof. The alter ego $\mathbb{S}_m$ from Example 3.3.1 yields a strong duality on $\mathsf{ISP}(\mathbf{S}_m)$. So the condition (IC) holds. Therefore, by Lemma 1.2.10, every compatible relation on $\mathbf{S}_m$ is equivalent to one of the form $\mathrm{E}(\mathbb{X}){\restriction}_S$, for some structure $\mathbb{X} \in \mathsf{ISP}_\mathsf{f}(\mathbb{S}_m)$ and non-empty generating set S for $\mathbb{X}$.

By Remark 4.3.3, such a structure $\mathbb{X}$ is the disjoint union of its d-connected substructures. Assume that $\mathbb{X}$ is not d-connected. We can write $\mathbb{X} = \mathbb{Z}_1 \,\dot{\cup}\, \mathbb{Z}_2$, where $\mathbb{Z}_1, \mathbb{Z}_2$ are substructures of $\mathbb{X}$. Then $T_1 := Z_1 \cap S$ is a non-empty generating set for $\mathbb{Z}_1$, and $T_2 := Z_2 \cap S$ is a non-empty generating set for $\mathbb{Z}_2$. We have

$$\begin{aligned} \mathrm{E}(\mathbb{X}){\restriction}_S &= \{\, \alpha{\restriction}_S \mid \alpha \colon \mathbb{Z}_1 \,\dot{\cup}\, \mathbb{Z}_2 \to \mathbb{S}_m \,\} \\ &\equiv \{\, \alpha_1{\restriction}_{T_1} \mid \alpha_1 \colon \mathbb{Z}_1 \to \mathbb{S}_m \,\} \times \{\, \alpha_2{\restriction}_{T_2} \mid \alpha_2 \colon \mathbb{Z}_2 \to \mathbb{S}_m \,\} \\ &= \mathrm{E}(\mathbb{Z}_1){\restriction}_{T_1} \times \mathrm{E}(\mathbb{Z}_2){\restriction}_{T_2}. \end{aligned}$$

Note that $\mathbb{X}_1$ embeds into $\mathbb{S}_m$ (from Lemma 4.3.1). There are two constant morphisms from Z_i to X_1, for $i \in \{1, 2\}$, respectively sending all the elements of Z_i to a_0 and 1 of X_1. Therefore the relations $\mathrm{E}(\mathbb{Z}_1){\restriction}_{T_1}$ and $\mathrm{E}(\mathbb{Z}_2){\restriction}_{T_2}$ are non-trivial. So the relation $\mathrm{E}(\mathbb{X}){\restriction}_S$ is not directly indecomposable.

Using Lemma 1.2.7, we now only need to find a finite upper bound on the number (up to equivalence) of relations $\mathrm{E}(\mathbb{X}){\restriction}_S$ where $\mathbb{X}$ is a d-connected structure in $\mathsf{ISP_f}(\mathbb{S}_m)$ and S a generating set for $\mathbb{X}$. Each d-connected structure in $\mathsf{ISP_f}(\mathbb{S}_m)$ has at most m $\ll$-connected components, by Remark 4.3.3. So, by Lemma 4.3.4, we can use the upper bound $m \times 8^m$. □

It follows from the previous lemma with $m = 1$ that the 3-element dual Stone algebra $\mathbf{S}_1$ admits only finitely many relations. This was claimed without proof in [24].

4.4 Ockham algebras with infinitely many relations

In this section, we shall check that the eight finite Ockham algebras whose dual spaces are given in Figure 4.2 each admit infinitely many relations. Using symmetry, there are only six algebras to consider. Two are already known to admit infinitely many relations for general reasons.

Lemma 4.4.1 ([24, 3.4])**.**

(1) *The dual of the Ockham space $\mathbb{Y}_1$ is the* 4*-element Boolean algebra, which admits infinitely many relations because it is the square of a non-trivial algebra.*

(2) *The dual of the Ockham space $\mathbb{Y}_4$ is the dual Stone algebra on the* 4*-element chain, which admits infinitely many relations because it has a pair of non-permuting congruences.*

So it remains to consider the Ockham algebras dual to $\mathbb{Y}_2$, $\mathbb{Y}_3$, $\mathbb{Y}_5$ and $\mathbb{Y}_6$. We shall be able to deal with these four algebras two at a time. To show that an algebra admits infinitely many relations we shall adapt the following technique from [24].

Lemma 4.4.2 ([24, 2.7])**.** *Let* $\mathbf{M}$ *be a finite algebra. To show that* $\mathbf{M}$ *admits infinitely many relations, it suffices to find*

- *an alter ego* $\mathbb{M}$ *of* $\mathbf{M}$*, and*
- *for each* $n \in \mathbb{N}$*, a structure* $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{M})$ *and a map* $\varphi_n \colon X_n \to M$ *that is not a morphism from* $\mathbb{X}_n$ *to* $\mathbb{M}$

such that, either for all $k < \ell$ *in* $\mathbb{N}$ *or for all* $k > \ell$ *in* $\mathbb{N}$*, the following holds:*

- *for each* $\omega \colon \mathbb{X}_\ell \to \mathbb{X}_k$*, the map* $\varphi_k \circ \omega$ *is a morphism from* $\mathbb{X}_\ell$ *to* $\mathbb{M}$*.*

In the proof of this lemma, the sequence of structures $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \dots$ in $\mathsf{ISP_f}(\mathbb{M})$ is used to define a sequence of compatible relations $r_1, r_2, r_3, \dots$ on $\mathbf{M}$, where $r_n := \mathrm{E}(\mathbb{X}_n){\restriction}_{X_n}$. The assumptions of the lemma are set up to ensure that these relations are pairwise non-equivalent.

We start by considering the Ockham space $\mathbb{Y}_3$, which is the dual of the 3-element Kleene algebra $\mathbf{K} = \langle\{0, a, 1\}; \vee, \wedge, f, 0, 1\rangle$ shown in Figure 4.4. Define the enriched ordered set $\mathbb{K}^\flat := \langle\{0, a, 1\}; \ll_0, K_0\rangle$ also shown in Figure 4.4, where $K_0 = \{0, 1\}$. Note that $\mathbb{K}^\flat$ is a reduct of the alter ego $\mathbb{K}$ of $\mathbf{K}$ given in Example 3.2.6. (In fact, the relations $\ll_0$ and K_0 determine the clone of $\mathbf{K}$; see [13, 4.3.12].) Rather than applying Lemma 4.4.2 directly to $\mathbf{K}$ and $\mathbb{K}^\flat$, we prove a more general result that will also cover the Ockham algebra with dual space $\mathbb{Y}_2$.

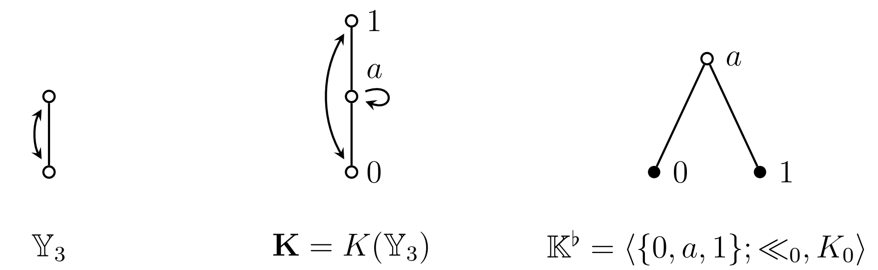


Figure 4.4: The 3-element Kleene algebra

Lemma 4.4.3. *Let* $\mathbf{M}$ *be a finite algebra and let* $\mathbb{M} = \langle M; r, s\rangle$ *be an alter ego of* $\mathbf{M}$*, where* $r \subseteq M^2$ *and* $s \subseteq M$*. Then* $\mathbf{M}$ *admits infinitely relations provided* $\mathsf{ISP_f}(\mathbb{M})$ *contains the enriched ordered set* $\mathbb{B} = \langle\{0, a, 1\}; \leqslant, \{0\}\rangle$ *shown in Figure 4.5.*

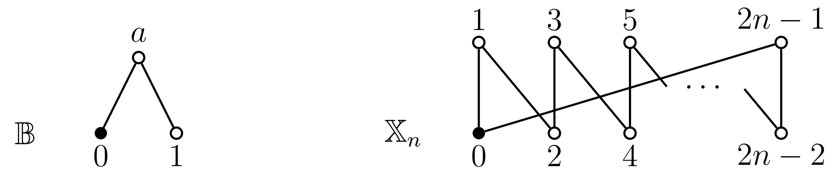


Figure 4.5: Structures for Lemma 4.4.3

Proof. Assume that $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$. We shall use Lemma 4.4.2 to show that $\mathbf{M}$ admits infinitely many relations. Let $n \in \mathbb{N}$ with $n \geqslant 2$ and define the structure $\mathbb{X}_n = \langle \{0, 1, \dots, 2n-1\}; \leqslant, s \rangle$ shown in Figure 4.5: the ordered set $\langle X_n; \leqslant \rangle$ is the $2n$-element crown and $s = \{0\}$. Now define the map $\psi_n \colon X_n \to B$ by

$$\psi_n(i) = \begin{cases} 1, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then ψ_n is not a morphism from $\mathbb{X}_n$ to $\mathbb{B}$, as n and $n+1$ are comparable in $\mathbb{X}_n$, but $\psi_n(n) = 1$ and $\psi(n+1) = 0$ are not comparable in $\mathbb{B}$. Since $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$, there must be a morphism $\rho_n \colon \mathbb{B} \to \mathbb{M}$ such that $\varphi_n := \rho_n \circ \psi_n$ is not a morphism from $\mathbb{X}_n$ to $\mathbb{M}$.

Using Lemma 4.4.2, the following two claims establish that $\mathbf{M}$ admits infinitely many relations.

Claim 1: *$\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{M})$, for all $n \geqslant 2$.*

Since $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$ by assumption, it is enough to show that $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{B})$. We shall use the Separation Theorem 1.3.9.

Let $x, y \in X_n$ with $x \nleqslant y$ in $\mathbb{X}_n$. If $x \neq 0$, then we can define the morphism $\alpha_x \colon \mathbb{X}_n \to \mathbb{B}$ by

$$\alpha_x(z) = \begin{cases} a, & \text{if } z \in {\uparrow}x, \\ 0, & \text{otherwise,} \end{cases}$$

and $\alpha_x(x) = a \nleqslant 0 = \alpha_x(y)$ in $\mathbb{B}$. If $x = 0$, then we can define the morphism

$\beta\colon \mathbb{X}_n \to \mathbb{B}$ by

$$\beta(z) = \begin{cases} 0, & \text{if } z = 0, \\ a, & \text{if } z = 1 \text{ or } z = 2n-1, \\ 1, & \text{otherwise,} \end{cases}$$

and $\beta(x) = 0 \nleqslant 1 = \beta(y)$ in $\mathbb{B}$. Thus the order $\leqslant$ is separated by morphisms from $\mathbb{X}_n$ to $\mathbb{B}$, and it follows that the elements of X_n are also separated. Finally, define the morphism $\gamma\colon \mathbb{X}_n \to \mathbb{B}$ by

$$\gamma(z) = \begin{cases} 0, & \text{if } z = 0, \\ a, & \text{otherwise.} \end{cases}$$

Then γ separates the unary relation s. Hence $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{B})$.

Claim 2: *Let $k, \ell \in \mathbb{N}$ with $2 \leqslant \ell < k$ and let $\omega\colon \mathbb{X}_\ell \to \mathbb{X}_k$. Then $\varphi_k \circ \omega$ is a morphism from $\mathbb{X}_\ell$ to $\mathbb{M}$.*

Since $\varphi_k := \rho_k \circ \psi_k$, it suffices to show that $\psi_k \circ \omega$ is a morphism from $\mathbb{X}_\ell$ to $\mathbb{B}$.

For any connected ordered set $\mathbb{P}$, there is a distance function $d\colon P^2 \to \mathbb{N} \cup \{0\}$ on $\mathbb{P}$, where $d(a, b)$ is the length of the shortest fence in $\mathbb{P}$ between a and b. We shall use the distance functions d_k and d_ℓ on $\mathbb{X}_k$ and $\mathbb{X}_\ell$.

We first show that $\omega(X_\ell) \subseteq X_k \backslash \{k\}$. Let $x \in X_\ell$. The 2ℓ-crown $\mathbb{X}_\ell$ has diameter ℓ, and so $d_\ell(0, x) \leqslant \ell$. Note that $\omega(0) = 0$, as ω preserves s. Since ω is order preserving, we have

$$\ell \geqslant d_\ell(0, x) \geqslant d_k(\omega(0), \omega(x)) = d_k(0, \omega(x)).$$

But $d_k(0, k) = k > \ell$, and therefore $\omega(x) \in X_k \backslash \{k\}$.

The restriction of the map ψ_k to $X_k \backslash \{k\}$ is constant 0, and therefore preserves both $\leqslant$ and s. Since $\omega(X_\ell) \subseteq X_k \backslash \{k\}$, it now follows that $\psi_k \circ \omega\colon \mathbb{X}_\ell \to \mathbb{B}$ is a morphism. □

Lemma 4.4.4. *The* 3*-element Kleene algebra* (*which is the Ockham algebra dual to* $\mathbb{Y}_3$) *admits infinitely many relations.*

Proof. This follows easily from Lemma 4.4.3 using the alter ego $\mathbb{K}^\flat$, where $\mathbb{K}^\flat = \langle\{0, a, 1\}; \ll_0, K_0\rangle$ shown in Figure 4.4. The structure $\mathbb{B}$ from Figure 4.5 embeds into $(\mathbb{K}^\flat)^2$ via $0 \mapsto (0, 0)$, $a \mapsto (a, a)$, $1 \mapsto (1, a)$. □

The dual of $\mathbb{Y}_2$ is the 4-element Ockham algebra $\mathbf{A}_2 = \langle\{0, a, b, 1\}; \vee, \wedge, f, 0, 1\rangle$ shown in Figure 4.6.

Lemma 4.4.5. *The Ockham algebra* $\mathbf{A}_2$ *with dual space* $\mathbb{Y}_2$ *admits infinitely many relations.*

Proof. Define the enriched ordered set $\mathbb{A}_2 = \langle\{0, a, b, 1\}; \preccurlyeq, s\rangle$ shown in Figure 4.6, where $s = \{0, 1\}$. Then $\mathbb{A}_2$ is an alter ego of $\mathbf{A}_2$; see the diagram of $\preccurlyeq$ as a subalgebra of $(\mathbf{A}_2)^2$ in Figure 4.6. The structure $\mathbb{B}$ from Lemma 4.4.3 embeds into $(\mathbb{A}_2)^2$ via $0 \mapsto (0, 1)$, $a \mapsto (a, 1)$, $1 \mapsto (a, b)$. So $\mathbf{A}_2$ admits infinitely many relations. □

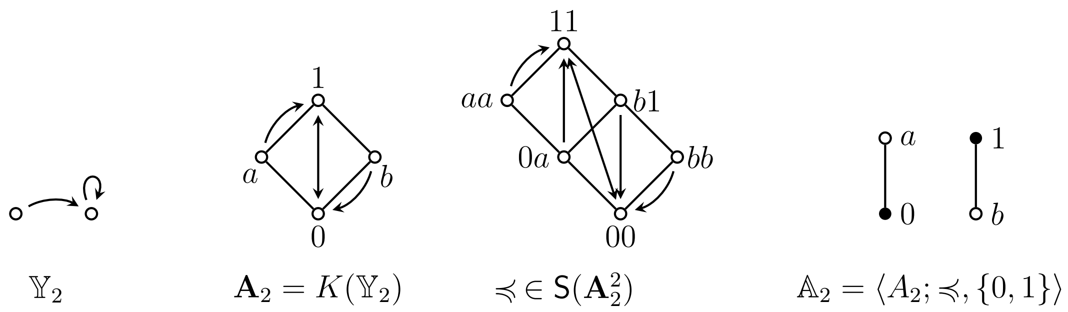


Figure 4.6: The Ockham algebra $\mathbf{A}_2$ with dual space $\mathbb{Y}_2$

Our second general lemma will cover the remaining two Ockham algebras $K(\mathbb{Y}_5)$ and $K(\mathbb{Y}_6)$.

Lemma 4.4.6. *Let* $\mathbf{M}$ *be a finite algebra and let* $\mathbb{M} = \langle M; r, s\rangle$ *be an alter ego of* $\mathbf{M}$, *where* $r, s \subseteq M^2$. *Then* $\mathbf{M}$ *admits infinitely relations provided* $\mathsf{ISP_f}(\mathbb{M})$ *contains the structure* $\mathbb{B} = \langle\{0, a, b, 1\}; \leqslant, \trianglelefteq\rangle$ *shown in Figure 4.7, with the order* $\leqslant = \Delta_B \cup \{(0, a), (1, b)\}$ *and the quasi-order* $\trianglelefteq = \leqslant \cup \{(a, 0)\}$.

Proof. Assume that $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$. We use Lemma 4.4.2 to show that $\mathbf{M}$ admits infinitely many relations. Let $n \in \mathbb{N}$ and define $\mathbb{X}_n = \langle\{0, 1, \dots, 2n\}; \leqslant, \trianglelefteq\rangle$ be the structure given in Figure 4.7: the ordered set $\langle X_n; \leqslant\rangle$ is the $(2n+1)$-element fence, and the quasi-order $\trianglelefteq$ on $\mathbb{X}_n$ is given by

$$\trianglelefteq = (L_n)^2 \cup (U_n)^2 \cup (L_n \times U_n),$$

where $L_n := \{0, 2n\}$ and $U_n := \{1, 2, \dots, 2n-1\}$. Define $\psi_n \colon X_n \to B$ by

$$\psi_n(x) = \begin{cases} 1, & \text{if } x = 2n, \\ b, & \text{otherwise.} \end{cases}$$

Then ψ_n is not a morphism from $\mathbb{X}_n$ to $\mathbb{B}$, as we have $(0, 2n) \in \trianglelefteq^{\mathbb{X}_n}$ but $(\varphi_n(0), \varphi_n(2n)) = (b, 1) \notin \trianglelefteq^{\mathbf{B}}$. Since $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$, there must be a morphism $\rho_n \colon \mathbb{B} \to \mathbb{M}$ such that $\varphi_n := \rho_n \circ \psi_n$ is not a morphism from $\mathbb{X}_n$ to $\mathbb{M}$.

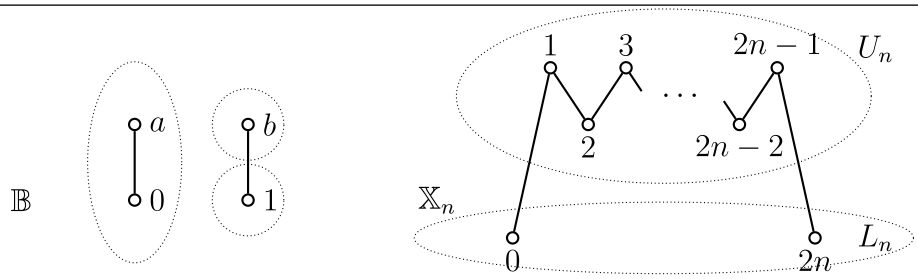


Figure 4.7: Structures for Lemma 4.4.6

Using Lemma 4.4.2, we can show that $\mathbf{M}$ admits infinitely many relations by establishing the following two claims.

Claim 1*:* $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{M})$*, for all* $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Since $\mathbb{B} \in \mathsf{ISP_f}(\mathbb{M})$ by assumption, it is enough to show that $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{B})$. Let $x, y \in X_n$ with $(x, y) \notin \leqslant^{\mathbb{X}_n}$. Since $\{0, a\}^2 \subseteq \trianglelefteq^{\mathbb{B}}$, we can define the morphism $\alpha_x \colon \mathbb{X}_n \to \mathbb{B}$ by

$$\alpha_x(z) = \begin{cases} a, & \text{if } x \leqslant^{\mathbb{X}_n} z, \\ 0, & \text{otherwise,} \end{cases}$$

and we have $(\alpha_x(x), \alpha_x(y)) = (a, 0) \notin \leqslant^{\mathbb{B}}$. Now let $x, y \in X_n$ with $(x, y) \notin \trianglelefteq^{\mathbb{X}_n}$ and define the map $\beta_x \colon X_n \to B$ by

$$\beta_x(z) = \begin{cases} b, & \text{if } x \trianglelefteq^{\mathbb{X}_n} z, \\ 1, & \text{otherwise.} \end{cases}$$

Then β_x preserves $\trianglelefteq$ and it follows that β_x preserves $\leqslant$, since $\leqslant^{\mathbb{X}_n} \subseteq \trianglelefteq^{\mathbb{X}_n}$ and $\trianglelefteq^{\mathbb{B}} \cap \{b, 1\}^2 \subseteq \leqslant^{\mathbb{B}}$. So $\beta_x \colon \mathbb{X}_n \to \mathbb{B}$ is a morphism, and $(\beta(x), \beta(y)) = (b, 1) \notin \trianglelefteq^{\mathbb{B}}$. Hence $\mathbb{X}_n \in \mathsf{ISP_f}(\mathbb{B})$, by the Separation Theorem 1.3.9.

Claim 2: Let $\ell < k$ in $\mathbb{N}$ and let $\omega \colon \mathbb{X}_\ell \to \mathbb{X}_k$. Then $\varphi_k \circ \omega$ is a morphism from $\mathbb{X}_\ell$ to $\mathbb{M}$.

The distance between the elements 0 and $2k$ in the ordered-set reduct of $\mathbb{X}_k$ is $2k$. Since the ordered-set reduct of $\mathbb{X}_\ell$ has diameter $2\ell < 2k$, it follows that $\{0, 2k\} \nsubseteq \omega(X_\ell)$. So $\omega(X_\ell) \subseteq X_k \backslash \{0\}$ or $\omega(X_\ell) \subseteq X_k \backslash \{2k\}$. From the definition of ψ_k, it is easy to see that both maps $\psi_k {\restriction}_{X_k \backslash \{0\}}$ and $\psi_k {\restriction}_{X_k \backslash \{2k\}}$ preserve $\leqslant$ and $\trianglelefteq$. Hence $\psi_k \circ \omega$ is a morphism from $\mathbb{X}_\ell$ to $\mathbb{B}$, and thus $\varphi_k \circ \omega = \rho_k \circ \psi_k \circ \omega$ is a morphism from $\mathbb{X}_\ell$ to $\mathbb{M}$. □

Lemma 4.4.7. *The Ockham algebras with dual spaces $\mathbb{Y}_5$ and $\mathbb{Y}_6$ admit infinitely many relations.*

Proof. The Ockham algebras $\mathbf{A}_5 := K(\mathbb{Y}_5)$ and $\mathbf{A}_6 := K(\mathbb{Y}_6)$ are shown in Figure 4.8. We take both algebras to have the same underlying set $A = \{0, a, b, 1\}$. Define the structure $\mathbb{A} = \langle A; \leqslant, \trianglelefteq \rangle$ as shown in Figure 4.8, with the order $\leqslant$ and quasi-order $\trianglelefteq$ given by

$$\leqslant\, = \Delta_A \cup \{(0, a), (1, b)\} \quad \text{and} \quad \trianglelefteq\, = \,\leqslant \cup \{(a, 0)\}$$

Then it is straightforward to check that $\mathbb{A}$ is an alter ego of both $\mathbf{A}_5$ and

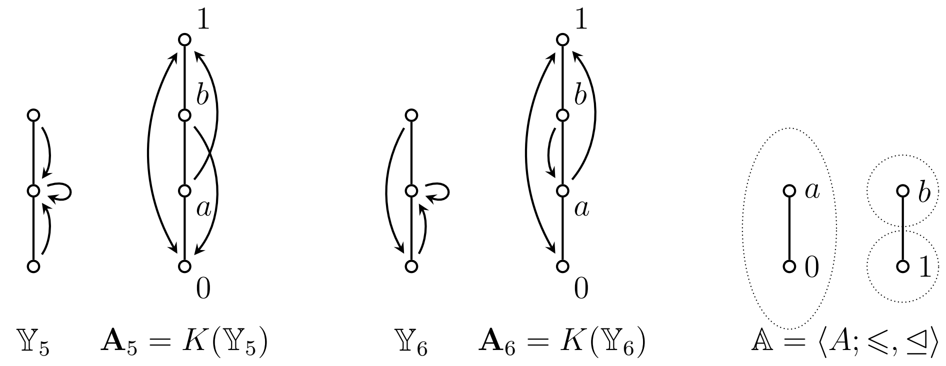


Figure 4.8: The Ockham algebras dual to $\mathbb{Y}_5$ and $\mathbb{Y}_6$

$\mathbf{A}_6$ (see Figure 4.9). So it follows immediately from Lemma 4.4.6 that both $\mathbf{A}_5$ and $\mathbf{A}_6$ admit infinitely many relations. □

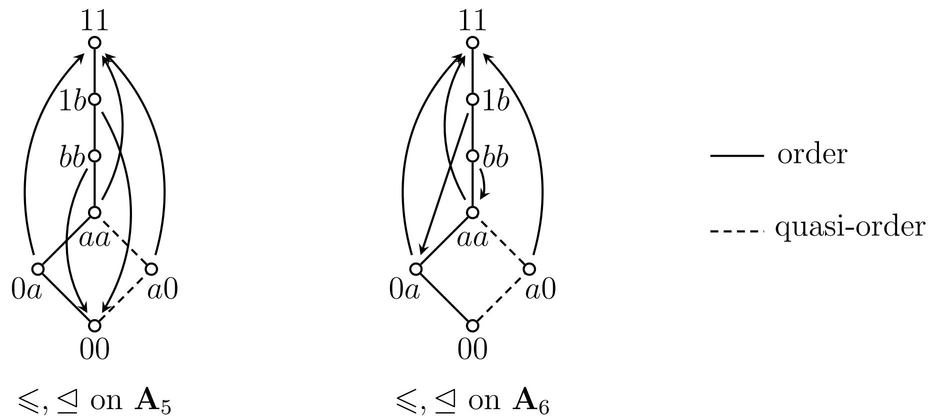


Figure 4.9: The order $\leqslant$ and quasi-order $\trianglelefteq$ on $\mathbf{A}_5$ and $\mathbf{A}_6$

For finite Ockham algebras $\mathbf{A}$ and $\mathbf{B}$ such that $H(\mathbf{A})$ is a divisor of $H(\mathbf{B})$, if $\mathbf{A}$ admits infinitely many relations, then $\mathbf{B}$ does too, by Lemmas 1.2.14 and 4.1.3. So the implication (1) $\Rightarrow$ (3) of Theorem 4.1.1 now follows from Lemmas 4.4.1, 4.4.4, 4.4.5 and 4.4.7.

4.5 Completing the proof

In this section, we shall complete the proof of Theorem 4.1.1 by showing that (3) ⇒ (2).

Lemma 4.5.1. *Let $\mathbb{X}$ be a non-empty finite Ockham space. Assume that $\mathbb{X}$ is not isomorphic to any of the Ockham spaces in Figure 4.1. Then one of the Ockham spaces in Figure 4.2 is a divisor of $\mathbb{X}$.*

Proof. Since $\mathbb{X} = \langle X; \leqslant, g\rangle$ is finite, it follows that $\mathbb{X}$ must contain an n-cycle, for some $n \in \mathbb{N}$. We break the proof up into three cases.

Case 1: $\mathbb{X}$ *contains an even cycle.*

The Ockham space $\mathbb{Y}_3$ is a divisor of $\mathbb{X}$, by Lemma 4.2.5.

Case 2: $\mathbb{X}$ *contains two different odd cycles.*

Let C and D be disjoint odd cycles of $\mathbb{X}$. Define $\mathbb{Z} \leqslant \mathbb{X}$ by $Z := C \cup D$. Then $\mathbb{Z}$ is an antichain, by Lemma 4.2.4. So we can define the morphism $\alpha \colon \mathbb{Z} \to \mathbb{Y}_1$ by

$$\alpha(z) = \begin{cases} 0, & \text{if } z \in C, \\ 1, & \text{if } z \in D. \end{cases}$$

Thus $\mathbb{Y}_1 \in \mathsf{HS}(\mathbb{X})$.

Case 3: $\mathbb{X}$ *contains only one cycle.*

Let C be the unique cycle of $\mathbb{X}$. By Case 1, we can assume that $m := |C|$ is odd. So C is an antichain in $\mathbb{X}$, by Lemma 4.2.4. Since $\mathbb{X}$ is not isomorphic to $\mathbb{C}_m$ from Figure 4.1, we must have $X \backslash C \neq \varnothing$. We consider two subcases.

Case 3a: $\mathbb{X}$ *is one-generated.*

There is $x \in X \backslash C$ with $g(x) \in C$. If $C \cup \{x\}$ is an antichain, then it is easy to see that $\mathbb{Y}_2 \in \mathsf{HS}(\mathbb{X})$. So we can assume without loss of generality that $x \geqslant c$, for some $c \in C$. Since g is order-reversing, we get $g(x) \leqslant g(c)$. But $g(x) \in C$ and so $g(x) = g(c)$, as C is an antichain. The substructure of $\mathbb{X}$ on $C \cup \{x\}$

is isomorphic to $\mathbb{D}_m$ from Figure 4.1. Therefore $X\backslash(C \cup \{x\}) \neq \varnothing$. Since $\mathbb{X}$ is one-generated, there is $y \in X\backslash(C \cup \{x\})$ with $g(y) = x$.

As C is an antichain, we must have $x \notin {\downarrow}C$ or $x \notin {\uparrow}C$. We are assuming that $x \in {\uparrow}C$ and so $x \notin {\downarrow}C$. Since g is order-reversing, it follows that $y \notin {\uparrow}(C \cup \{x\})$. Now define $\mathbb{Z} \leqslant \mathbb{X}$ by $Z := C \cup \{x, y\}$. We can define the morphism $\beta\colon \mathbb{Z} \to \mathbb{Y}_6$ by

$$\beta(z) = \begin{cases} 2, & \text{if } z = x, \\ 1, & \text{if } z \in C, \\ 0, & \text{if } z = y, \end{cases}$$

and therefore $\mathbb{Y}_6 \in \mathsf{HS}(\mathbb{X})$.

Case 3b: $\mathbb{X}$ *is not one-generated.*

There are distinct $x, y \in X\backslash C$ such that $g(x), g(y) \in C$. Define $\mathbb{Z} \leqslant \mathbb{X}$ by $Z := C \cup \{x, y\}$. If $x \notin {\downarrow}C$ and $y \notin {\downarrow}C$, then without loss of generality $x \nleqslant y$ and we can define $\gamma\colon \mathbb{Z} \to \mathbb{Y}_4$ by

$$\gamma(z) = \begin{cases} 2, & \text{if } z = x, \\ 1, & \text{if } z = y, \\ 0, & \text{if } z \in C, \end{cases}$$

and so $\mathbb{Y}_4 \in \mathsf{HS}(\mathbb{X})$. Similarly, if $x \notin {\uparrow}C$ and $y \notin {\uparrow}C$, then we can show that $\mathbb{Y}_4^\partial \in \mathsf{HS}(\mathbb{X})$.

Without loss of generality, we can now assume that $x \notin {\uparrow}C$ and $y \notin {\downarrow}C$. In this case, it is easy to check that $\mathbb{Y}_5 \in \mathsf{HS}(\mathbb{X})$. □

This completes the proof of Theorem 4.1.1.

Remark 4.5.2. Our characterisation for Ockham algebras in general can easily be restricted to yield characterisations within familiar subvarieties. For example, we noted in Remark 3.1.1 that the variety $\boldsymbol{\mathcal{MS}}$ of MS-algebras consists of all Ockham algebras with dual spaces satisfying $x \leqslant g^2(x)$. The only dual spaces in Figure 4.1 that satisfy this are $\mathbb{C}_1$ and $\mathbb{D}_1^\partial$. So the only non-trivial MS algebras with only finitely many relations are the 2-element Boolean algebra and the 3-element Stone algebra. Using this technique, we obtain the following characterisations for the other subvarieties from Remark 3.1.1:

- a finite Kleene algebra **K** admits only finitely many relations if and only if $|K| \leqslant 2$;
- a finite De Morgan algebra **M** admits only finitely many relations if and only if $|M| \leqslant 2$;
- a finite Stone algebra **S** admits only finitely many relations if and only if $|S| \leqslant 3$.

Conclusion

In this thesis, our aim was to expand the set of examples of finite algebras with only finitely many compatible relations, up to interdefinability. We characterised all finite algebras that admit only finitely many relations within two familiar classes of lattice-based algebras: the class of Heyting algebras and the class of Ockham algebras. We used different approaches for these two classes.

We proved a general sufficient condition for a finite algebra $\mathbf{M}$ to admit only finitely many relations, namely that the alter ego $\mathbb{M} = \langle M; \mathrm{End}_{\mathrm{p}}(\mathbf{M})\rangle$ of $\mathbf{M}$ satisfies (IC) and is linear (Theorem 2.3.5). We then showed that finite Heyting chains satisfy this sufficient condition (Lemmas 2.1.9 and 2.3.6) and therefore admit only finitely many relations. This result and the result from the paper by Davey and Pitkethly [24] (see Example 1.2.15) lead to the characterisation for Heyting algebras: a finite Heyting algebra admits only finitely many relations if and only if it is a chain.

Using the restricted Priestley duality for Ockham algebras and piggyback dualities for a family of finite subdirectly irreducible Ockham algebras, introduced in Chapter 3, we characterised the finite Ockham algebras that admit only finitely many relations. The characterisation is stated in terms of restricted Priestley dual spaces (Theorem 4.1.1). Up to isomorphism and symmetry, there are two countably infinite families of Ockham algebras having this finiteness property. We also characterised the finiteness condition for several familiar subvarieties of the variety of Ockham algebras: Stone algebras, Kleene algebras, De Morgan algebras and MS algebras (Remark 3.1.1). Surprisingly, amongst all the finite algebras within these subvarieties, the two-element Boolean algebra and the three-element Stone algebra are the only non-trivial finite algebras that admit only finitely many relations.

The results above seem to suggest that not many finite algebras satisfy this finiteness condition. The techniques used in this thesis suggest that a general characterisation could be very difficult.

There are many other classes of lattice-based algebras that have not been studied in this thesis, such as the class of double Stone algebras. It would be interesting to investigate if the techniques introduced and developed in this thesis and in [24] could be used to characterise the finiteness condition within these classes. The techniques we have been using may not be enough for solving the problem within such classes of algebras, as the alter egos associated with these algebras could be much more complicated. Nevertheless, as double Stone algebras are closely related to Stone algebras, it seems reasonable to conjecture that the characterisation may be possible for this class.

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