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The Thrackle Conjecture for K_5 and $K_{3,3}$

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ABSTRACT. We prove the thrackle conjecture for K_5 and $K_{3,3}$. To do this we reduce the problem to a set of simultaneous quadratic equations over $\mathbb{Z}_2$. Parts of this proof are computer assisted.

1. Introduction

Let G be a finite graph with no loops or multiple edges. A *thrackle drawing* of G in a surface M is a drawing $\mathcal{T} : G \to M$ for which the edges are represented by Jordan arcs, such that each pair of edges meets precisely once, either at a vertex or at a proper crossing. John H. Conway's celebrated thrackle conjecture is that for thrackle drawings in the plane, one should have $\#edges \le \#vertices$ (see [**17, 18, 9, 14, 16, 11, 7, 13**]). More generally, for thrackles in an orientable surface M_g of genus g, one may "presumably expect the appropriate conjecture to be that max($\#edges - \#vertices$) depends on the genus of the surface"[**8**, Sect. F16]. The obvious conjecture is:

THRACKLE CONJECTURE. [**5**] *If $\mathcal{T} : G \to M_g$ is a thrackle, then $\#edges \le \#vertices + 2g$.*

In this paper we establish the following:

THEOREM 1.1. *The thrackle conjecture holds for K_5 and $K_{3,3}$.*

The motivation of this paper is that it gives a case study of the problems that arise when one approaches the thrackle conjecture for specific graphs. The above theorem also gives the following:

COROLLARY 1.2. *The thrackle conjecture holds for all graphs with ≤ 5 vertices, and for all bipartite graphs with ≤ 6 vertices.*

A thrackle drawing is determined by three pieces of information: 1) its rotation system (i.e., the set of rotation diagrams at the vertices), 2) the order of the crossings along each edge, and 3) the orientation of each crossing. Consider K_5 for example; here there are 5 vertices, 10 edges and 3 crossings on each edge giving a total of 15 crossings. At first sight, there would appear to be 6^5 choices for the

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rotation systems (see Section 2), 6^{10} choices for the orders of crossings, and 2^{15} choices for the orientations of the crossings. In total, this gives $\sim 10^{15}$ possibilities. For $K_{3,3}$, the figure is $\sim 4 \times 10^{19}$. For each such choice, there is a thrackle drawing with the chosen properties, and the drawing is unique, up to homeomorphism and the removal of handles disjoint from the drawing, and the underlying surface is similarly unique. In particular, the minimal genus of the surface is some (presently unknown) function of the finite combinatorial choices. The problem is to show that in none of these cases can the underlying surface have genus less than 3 for thrackles of K_5, or genus less than 2 for thrackles of $K_{3,3}$.

We begin in Section 2 by examining the rotation systems. Restricting one's attention to the rotation systems is the same as studying cellular embeddings rather than thrackles. Although there may seem at first sight to be a large number of possible rotation systems (6^5 for K_5 and 2^6 for $K_{3,3}$), the automorphisms group of the graph acts by change of vertex label, and elements in the same orbit give equivalent embeddings. This enables us to classify the cellular embeddings of K_5 and $K_{3,3}$ and hence reduce our study to a smaller number of rotation systems; 50 for K_5 and only 3 for $K_{3,3}$; see Proposition 2.1.

For thrackle drawings we can further reduce the possibilities for the rotation systems. In Section 3, for a thrackle drawing $\mathcal{T} : G \to M_g$ on a compact oriented surface M_g of genus g, we consider the $\mathbb{Z}_2$-intersection form Ω_{M_g} on the first homology group $\mathrm{H}_1(M_g)$, with values in $\mathbb{Z}_2$. Pulling back by $\mathcal{T}$, this defines a skew-symmetric form $\Omega_\mathcal{T} = \mathcal{T}^*\Omega_{M_g}$ on $\mathrm{H}_1(G)$, with the following obvious property:

$$\operatorname{rank}\Omega_\mathcal{T} \leq \operatorname{rank}\Omega_{M_g} = 2g.$$

Thus if G can be thrackled on a surface of genus g, one would have $\operatorname{rank}\Omega_\mathcal{T} \leq 2g$. The key point is that the form $\Omega_\mathcal{T}$ is entirely determined by the rotation system (see Lemma 3.1). We show that for (hypothetical) thrackles of $K_{3,3}$ and K_5 on surfaces of genus 1 and 2 respectively, there is at most one possible rotation system for $K_{3,3}$, and at most 2 for K_5; see Lemmas 3.3 and 3.4. So it remains to exclude these possibilities. Notice that a priori, at this point there still remain $\sim 4 \times 10^{12}$ possibilities for K_5 and $\sim 7 \times 10^{17}$ possibilities for $K_{3,3}$. But the main problem is not just that one is confronted by excessively long calculations, but rather that it is something of a notational nightmare to even write down the conditions imposed by the thrackle drawing in terms of the orders of the crossings on each edge and their orientations. The task is to find a subsystem of the set of thrackle conditions which is small enough to be written down compactly, but which is large enough so that its solution set is empty. The general principal we use in this paper is to only use conditions that depend on the order of the edge intersections, and not on the orientation of these crossings. It seems interesting that this strategy proves to be sufficient, and one may wonder if this reflects a general feature of thrackle drawings.

In Section 4, we consider K_5, and use the fact that in each of the two cases provided by Lemma 3.4, there is a null-homologous 5-cycle, which is itself a thrackle. From a general consideration of null-homologous curves (Theorem 4.1), we see that there are only two possibilities for the orders of intersections of a null-homologous thrackle of the 5-cycle (see Lemma 4.2). We use the knowledge of these orders to derive a system of quadratic equations over $\mathbb{Z}_2$; we show that this system has empty solution set, except in two special cases, which require a more subtle analysis. Similarly, in Section 5, we use the fact that in the sole case provided for $K_{3,3}$ by Lemma 3.3, there are 3 null-homologous 6-cycles, and that there are 256 possibilities for the orders of intersections of a null-homologous thrackle of the 6-cycle (see Lemma 5.1). Here our choice of system of equations is more computationally demanding than the K_5 case, but fortunately it can be dealt with without the treatment of special cases. Corollary 1.2 is proved in Section 6.

We performed many of the computations in this paper using Mathematica.

2. Embeddings of K_5 and $K_{3,3}$

In this section, we consider cellular embeddings of K_5 and $K_{3,3}$ in compact orientable surfaces; that is, embeddings for which the faces are homeomorphic to discs. Such an embedding is determined by its rotation system; that is the set of rotation diagrams at the vertices. Consider the general case of a graph Γ, with n vertices labelled 1 to n. At a vertex v of index k, a rotation diagram is a cyclic ordering of the k edges leaving v; so the set R_v of rotation diagrams at v may be identified with the set of equivalence classes on the symmetric group S_k, where two permutations are regarded as being equivalent if one can be obtained from the other by pre-multiplying by some power of the cyclic permutation $\rho = (1, 2, 3, \dots, k)$. Thus R_v may be regarded as the set of right cosets of the cyclic subgroup generated by ρ; so as a set, $R_v \cong S_{k-1}$. Thus for K_5, there are at first sight 6^5 cellular embeddings, while for $K_{3,3}$ there are 2^6. In fact, the number of inequivalent cellular embeddings is much lower, as one can see by considering the automorphism group of the given graph. For K_5, the automorphism group is S_5, and the rotation system has the form

$$RS(K_5) = R_1 \times R_2 \times R_3 \times R_4 \times R_5,$$

and S_5 acts on $RS(K_5)$ by changing the labels; it permutes the factors and changes the entries in the obvious way. For example, the transposition $(12) \in S_5$ sends the rotation system

$$r = \{(2,3,4,5),\ (3,4,5,1),\ (4,5,1,2),\ (5,1,2,3),\ (1,2,3,4)\}$$

to the new system:

$$\{(3,4,5,2),\ (1,3,4,5),\ (4,5,2,1),\ (5,2,1,3),\ (2,1,3,4)\}.$$

The number of inequivalent cellular embeddings can be further reduced by considering the change in orientation, which we can think of as the map τ on RS which reverses the order of each entry (without changing the order of the factors); for example, τ sends r to the new system:

$$\tau(r) = \{(2,5,4,3),\ (1,5,4,3),\ (2,1,5,4),\ (3,2,1,5),\ (4,3,2,1)\}.$$

This extends the action of S_5 on $RS(K_5)$ to an action of $S_5 \times \mathbb{Z}_2$.

For $K_{3,3}$, we label the vertices from 1 to 6, with the edges joining odd and even vertices. The rotation system is

$$RS(K_{3,3}) = R_1 \times R_2 \times R_3 \times R_4 \times R_5 \times R_6,$$

where each factor has only two elements; here is a typical element:

$$\{(2,4,6),\ (1,3,5),\ (2,4,6),\ (1,3,5),\ (2,4,6),\ (1,3,5)\}.$$

The automorphism group is $S_3 \times S_3 \times \mathbb{Z}_2$; the first factor acts by permuting the odd factors and changing the entries of the even factors, the second factor acts by permuting the even factors and changing the entries of the odd factors, and the third factor is generated by the element which acts on the factors and their

permutation type	**# elements**	# Fix_g	$\sum$# Fix_g
(12345)	24	6	144
(1234)	30	12	360
(123)	20	0	0
(12)	10	0	0
(123)(45)	20	0	0
(12)(34)	15	72	1080
id	1	7776	7776
τ(12345)	24	0	0
τ(1234)	30	0	0
τ(123)	20	0	0
τ(12)	10	48	480
τ(123)(45)	20	0	0
τ(12)(34)	15	144	2160
τ	1	0	0
total	240		12,000

TABLE 1. Action on rotation systems of K_5

permutation type	**# elements**	# Fix_g	$\sum$# Fix_g
(123)(456)	4	4	16
(123)(456)	4	16	64
(12)(45)	9	0	0
(12)	6	0	0
(123)(45)	12	0	0
id	1	64	64
(142536)	12	2	24
(1425)(36)	18	0	0
(14)(25)(36)	6	8	48
total	72		216

TABLE 2. Action on rotation systems of $K_{3,3}$

elements by the permutation (12)(34)(56). One could also consider the change of orientation τ, but for $K_{3,3}$ this turns out not to give any improvement.

We will use the so-called *Burnside's counting theorem* [**12**]: If a finite group G acts on a finite set, then #orbits $= \frac{1}{|G|}\sum_{g\in G}\mathrm{Fix}(g)$. Applying this to the above action of $S_5 \times \mathbb{Z}_2$ on $RS(K_5)$, one finds that there are $12,000/240 = 50$ orbits (see Table 1), while for the $S_3 \times S_3 \times \mathbb{Z}_2$ on $RS(K_{3,3})$, one finds that there are $216/72 = 3$ orbits (see Table 2). It is a simple matter to compute representatives of each orbit. This gives:

PROPOSITION 2.1. *There are 50 inequivalent cellular embeddings of K_5, and 3 inequivalent cellular embeddings of $K_{3,3}$; they are listed in Tables 3 and 4 respectively.*

1	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 1, 2, 3),	(1, 2, 3, 4)
2	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 1, 2, 3),	(1, 3, 2, 4)
3	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 1, 2, 3),	(2, 1, 3, 4)
4	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 1, 2, 3),	(2, 3, 1, 4)
5	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 1, 2, 3),	(3, 2, 1, 4)
6	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 2, 1, 3),	(1, 3, 2, 4)
7	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 2, 1, 3),	(2, 3, 1, 4)
8	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 2, 1, 3),	(3, 1, 2, 4)
9	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(5, 2, 1, 3),	(3, 2, 1, 4)
10	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 5, 2, 3),	(2, 1, 3, 4)
11	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 5, 2, 3),	(2, 3, 1, 4)
12	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 5, 2, 3),	(3, 1, 2, 4)
13	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 5, 2, 3),	(3, 2, 1, 4)
14	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(2, 1, 3, 4)
15	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(2, 3, 1, 4)
16	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(3, 2, 1, 4)
17	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(2, 5, 1, 3),	(2, 1, 3, 4)
18	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 5, 1, 2),	(2, 1, 5, 3),	(3, 2, 1, 4)
19	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(1, 3, 2, 4)
20	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(2, 3, 1, 4)
21	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(3, 1, 2, 4)
22	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(3, 2, 1, 4)
23	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 2, 1, 3),	(1, 3, 2, 4)
24	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 2, 1, 3),	(3, 1, 2, 4)
25	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(5, 2, 1, 3),	(3, 2, 1, 4)
26	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(1, 2, 5, 3),	(1, 3, 2, 4)
27	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(1, 2, 5, 3),	(2, 3, 1, 4)
28	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(1, 2, 5, 3),	(3, 1, 2, 4)
29	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(1, 2, 5, 3),	(3, 2, 1, 4)
30	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(2, 5, 1, 3),	(2, 3, 1, 4)
31	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(2, 5, 1, 3),	(3, 1, 2, 4)
32	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(2, 5, 1, 3),	(3, 2, 1, 4)
33	(2, 3, 4, 5),	(3, 4, 5, 1),	(4, 1, 5, 2),	(2, 1, 5, 3),	(3, 1, 2, 4)
34	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(5, 1, 2, 3),	(3, 1, 2, 4)
35	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(5, 1, 2, 3),	(3, 2, 1, 4)
36	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(5, 2, 1, 3),	(3, 1, 2, 4)
37	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(1, 5, 2, 3),	(3, 2, 1, 4)
38	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(1, 2, 5, 3),	(3, 2, 1, 4)
39	(2, 3, 4, 5),	(3, 4, 5, 1),	(5, 4, 1, 2),	(2, 1, 5, 3),	(3, 1, 2, 4)
40	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(2, 1, 3, 4)
41	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(2, 3, 1, 4)
42	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 5, 1, 2),	(1, 2, 5, 3),	(3, 2, 1, 4)
43	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 5, 1, 2),	(2, 5, 1, 3),	(2, 1, 3, 4)
44	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 5, 1, 2),	(2, 5, 1, 3),	(3, 2, 1, 4)
45	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(2, 1, 3, 4)
46	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 1, 5, 2),	(5, 1, 2, 3),	(2, 3, 1, 4)
47	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 1, 5, 2),	(2, 5, 1, 3),	(2, 3, 1, 4)
48	(2, 3, 4, 5),	(3, 5, 4, 1),	(4, 1, 5, 2),	(2, 1, 5, 3),	(3, 1, 2, 4)
49	(2, 3, 4, 5),	(3, 5, 4, 1),	(5, 1, 4, 2),	(2, 1, 5, 3),	(3, 1, 2, 4)
50	(2, 3, 4, 5),	(4, 5, 3, 1),	(5, 4, 1, 2),	(5, 2, 1, 3),	(3, 2, 1, 4)

TABLE 3. Cellular Embeddings of K_5

1	(2,4,6),	(1,3,5),	(2,4,6),	(1,3,5),	(2,4,6),	(1,3,5)
2	(2,6,4),	(1,3,5),	(2,4,6),	(1,3,5),	(2,4,6),	(1,3,5)
3	(2,6,4),	(1,5,3),	(2,4,6),	(1,3,5),	(2,4,6),	(1,3,5)

TABLE 4. Cellular Embeddings of $K_{3,3}$

3. Homology of the graph

For a thrackle drawing $\mathcal{T} : G \to M_g$ on a compact oriented surface M_g of genus g, we consider the $\mathbb{Z}_2$-intersection form Ω_{M_g} on the first homology group $\mathrm{H}_1(M_g)$, with values in $\mathbb{Z}_2$; if γ_1 and γ_2 are closed curves in M_g which intersect in a finite number k of transverse crossings, then $\Omega_{M_g}(\gamma_1, \gamma_2) = k \mod 2$. Pulling back by $\mathcal{T}$, this defines a skew-symmetric form $\Omega_{\mathcal{T}} = \mathcal{T}^*\Omega_{M_g}$ on $\mathrm{H}_1(G)$. We will be interested in the rank and kernel of this form. To compute $\Omega_{\mathcal{T}}$, note that the rotation system defined by $\mathcal{T}$ determines a cellular embedding $\mathcal{T}' : G \to M'$, into some compact orientable surface M'. Let $\sigma_{\mathcal{T}}$ denote the pull back by $\mathcal{T}'$ of the $\mathbb{Z}_2$-intersection form $\Omega_{M'}$ on $\mathrm{H}_1(M')$. Let l denote the $\mathbb{Z}_2$-length function on the 1-chain complex of G; that is, given a path c in G, $l(c)$ is the number mod(2) of edges in c. The following result was given in [**5**]:

LEMMA 3.1. *Suppose that $\mathcal{T} : G \to M_g$ is a thrackle drawing, and that c_1 and c_2 are cycles in G. Then $\Omega_{\mathcal{T}}(c_1, c_2) = \sigma_{\mathcal{T}}(c_1, c_2) + l(c_1).l(c_2) + l(c_1 \cap c_2) \pmod 2$. In particular, $\Omega_{\mathcal{T}}$ is entirely determined by the rotation system of $\mathcal{T}$.*

Let us first consider the case of $K_{3,3}$. For an arbitrary connected graph G, one has $\dim \mathrm{H}_1(G) = 1 + \#edges - \#vertices$. In particular, $\dim \mathrm{H}_1(K_{3,3}) = 4$ and so $\mathrm{H}_1(K_{3,3})$ contains $2^4 = 16$ elements. In fact, apart from the identity element, $\mathrm{H}_1(K_{3,3})$ is composed of nine 4-cycles and six 6-cycles. Let us label the vertices of $K_{3,3}$ with the numerals $1, \ldots, 6$ such that the edges join the odd vertices to even vertices (see Fig. 1). Then it is easy to verify that the 4-cycles $c_1 = 1234$, $c_2 = 1456$, $c_3 = 3456$ and $c_4 = 2345$ are linearly independent and hence form a basis for $\mathrm{H}_1(K_{3,3})$. Relative to this basis, the 4-cycles are:

$$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1),$$
$$(0,0,1,1), (0,1,1,0), (1,0,0,1), (1,1,0,1), (1,1,1,0),$$

and the 6-cycles are:

$$(1,1,0,0), (1,0,1,1), (0,1,1,1), (0,1,0,1), (1,0,1,0), (1,1,1,1).$$

Let us consider the rotation diagrams for $K_{3,3}$. At each vertex, there are 3 incident edges and so there are only 2 possible rotation diagrams at each vertex. At each odd vertex, the 3 edges terminate at the vertices $2, 4, 6$ respectively; for $i = 1, 3, 5$, we will set $o_i = 0$ if the anti-clockwise order of the edges incident to vertex i have terminal vertices $2, 4, 6$, and we set $o_i = 1$ otherwise; so $o_i = 1$ when the order of terminal vertices is an odd permutation of $2, 4, 6$. Similarly, for $i = 2, 4, 6$, we will set $o_i = 0$ if the anti-clockwise order of the edges incident to vertex i have terminal vertices in the cyclic order $1, 3, 5$, and we set $o_i = 1$ otherwise. Consider the matrix

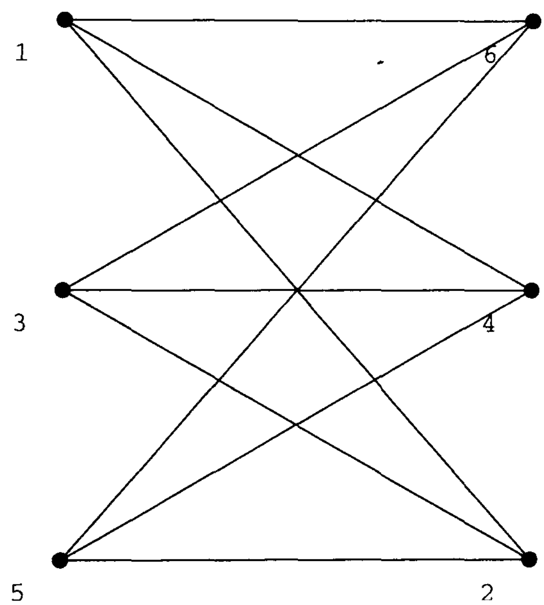

FIGURE 1

$\Omega_{i,j} = \Omega_{\mathcal{T}}(c_i, c_j)$. Using Lemma 3.1, one finds:

$$\Omega = \begin{pmatrix} 0 & o_1 + o_4 + 1 & o_3 + o_4 & o_2 + o_4 + 1 \\ o_1 + o_4 + 1 & 0 & o_4 + o_6 + 1 & o_4 + o_5 \\ o_3 + o_4 & o_4 + o_6 + 1 & 0 & o_3 + o_5 + 1 \\ o_2 + o_4 + 1 & o_4 + o_5 & o_3 + o_5 + 1 & 0 \end{pmatrix}.$$

If there exists a thrackle drawing of $K_{3,3}$ on the torus $\mathbb{T}$, then as $\dim \mathrm{H}_1(\mathbb{T}) = 2$, one has $\operatorname{rank} \Omega_{\mathbf{T}} = 2$. We consider rotation systems for which $\operatorname{rank} \Omega \leq 2$; that is, since Ω has even rank, $\operatorname{rank} \Omega \neq 4$, or equivalently, $\det \Omega = 0$. In fact, since Ω is skew-symmetric, the condition $\det \Omega = 0$ is equivalent to the condition $\mathrm{Pf}(\Omega) = 0$, where Pf denotes the Pfaffian (see for example [**15**]):

$$\mathrm{Pf}(\Omega) = \Omega_{1,2}\Omega_{3,4} - \Omega_{1,3}\Omega_{2,4} + \Omega_{1,4}\Omega_{2,3} \mod 2.$$

We will also make use of the following lemma from [**5**]:

LEMMA 3.2. *Suppose that $\mathcal{T} : G \to M_g$ is a thrackle drawing on any compact oriented surface M_g.*

(a) *If $c \subset G$ is a 4-cycle, then $\mathcal{T}(c)$ is nontrivial in $\mathbb{Z}_2$-homology.*
(b) *If $c_1, c_2 \subset G$ are 3-cycles, then $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ are not $\mathbb{Z}_2$-homologous.*

$K_{3,3}$ has no 3-cycles, but part (a) of Lemma 3.2 implies that the kernel of Ω contains no 4-cycles. We now consider the 3 rotation systems of Table 4. The first system has $o_1 = \cdots = o_6 = 0$, the second $o_1 = 1, o_2 = \cdots = o_6 = 0$, and the third $o_1 = o_2 = 1, o_3 = \cdots = o_6 = 0$. One finds that $\mathrm{Pf}(\Omega) \neq 0$ in the second case, while in the third case, Ω has the 4-cycle c_1 in its kernel. In the first case, the kernel consists of the identity element and the three 6-cycles whose coordinates are: $(0,1,0,1), (1,0,1,0), (1,1,1,1)$. These are the cycles 123654, 165234 and 125436 respectively. So one has:

LEMMA 3.3. *Of the 3 inequivalent rotation systems of $K_{3,3}$ listed in Table 4, only the first one,*

$$(2,4,6),\ (1,3,5),\ (2,4,6),\ (1,3,5),\ (2,4,6),\ (1,3,5),$$

has rank $\Omega_{\mathcal{T}} \leq 2$ *and has the property that its 4-cycles are all nonzero in homology. In this case,*

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We now turn to K_5. One has $\dim \mathrm{H}_1(K_5) = 6$ and so $\mathrm{H}_1(K_5)$ contains $2^6 = 64$ elements. An (ordered) basis $c_1, \dots, c_6$ is given by the 5-cycle $c_1 = 12345$ and the 5 3-cycles: $c_2, \dots, c_6 = 123, 234, 345, 451, 512$ respectively. Apart from the identity element, $\mathrm{H}_1(K_5)$ is comprised of 10 3-cycles, 15 4-cycles, 12 5-cycles, 15 sums of 2 3-cycles, 10 sums of a 3-cycle and 4-cycle, and K_5 itself. Moreover, there is a duality on $\mathrm{H}_1(K_5)$ determined by taking the complement in K_5. Relative to the given basis, K_5 itself is the element $(1,1,1,1,1)$, so to form the dual of an element, one just replaces the 0's by 1's and visa-versa. Given the above, it suffices to describe the $3, 4$ and 5-cycles. Relative to the given basis, the 3-cycles are:

$$c_2, \dots, c_6, c_1 + c_3 + c_6, c_1 + c_2 + c_4, c_1 + c_3 + c_5, c_1 + c_4 + c_6, c_1 + c_2 + c_5,$$

the 4-cycles are:

$$c_1 + c_2, c_1 + c_3, c_1 + c_4, c_1 + c_5, c_1 + c_6, c_2 + c_3, c_3 + c_4, c_4 + c_5, c_5 + c_6, c_2 + c_6,$$
$$c_1 + c_2 + c_4 + c_5, c_1 + c_3 + c_5 + c_6, c_1 + c_2 + c_4 + c_6, c_1 + c_2 + c_3 + c_5, c_1 + c_3 + c_4 + c_6,$$

and the 5-cycles are:

$$c_1, c_1 + c_2 + c_6, c_1 + c_2 + c_3, c_1 + c_3 + c_4, c_1 + c_4 + c_5, c_1 + c_5 + c_6,$$

and their duals.

We now consider the rotation systems for K_5. At each vertex, there are 4 incident edges and so there are 6 possible rotation diagrams at each vertex. For each vertex i, we introduce three $\mathbb{Z}_2$-valued functions e_i, e_i^+, e_i^- on the set R_i of rotation diagram at vertex i. They are defined as follows: we label the edges incident to vertex i by their terminal vertices. Then we set $e_i^+ = 0$ if the anti-clockwise cyclic order of the edges $1+m, 2+m, 4+m \pmod 5$ is a positive permutation of these number, and we set $e_i^+ = 1$ otherwise. Similarly, $e_i^- = 0$ if the anti-clockwise cyclic order of the edges $1+m, 3+m, 4+m \pmod 5$ is a positive permutation, and $e_i^- = 1$ otherwise. And $e_i = 0$ if in the anti-clockwise cyclic order of the edges $1+m, 2+m, 3+m, 4+m \pmod 5$, the edges $1+m, 2+m \pmod 5$ are adjacent, and $e_i = 1$ otherwise. Now consider the intersection form relative to the basis $c_1, \dots, c_6$: this is the skew symmetric matrix $\Omega_{i,j} = \Omega_{\mathcal{T}}(c_i, c_j)$. Using Lemma 3.1, one finds:

$$\Omega = \begin{pmatrix} 0 & e_3^- + e_1^+ + 1 & e_4^- + e_2^+ + 1 & e_5^- + e_3^+ + 1 & e_1^- + e_4^+ + 1 & e_2^- + e_5^+ + 1 \\ e_3^- + e_1^+ + 1 & 0 & e_3^- + e_2^+ + 1 & e_3 + 1 & e_1 + 1 & e_2^- + e_1^+ + 1 \\ e_4^- + e_2^+ + 1 & e_3^- + e_2^+ + 1 & 0 & e_4^- + e_3^+ + 1 & e_4 + 1 & e_2 + 1 \\ e_5^- + e_3^+ + 1 & e_3 + 1 & e_4^- + e_3^+ + 1 & 0 & e_5^- + e_4^+ + 1 & e_5 + 1 \\ e_1^- + e_4^+ + 1 & e_1 + 1 & e_4 + 1 & e_5^- + e_4^+ + 1 & 0 & e_1^- + e_5^+ + 1 \\ e_2^- + e_5^+ + 1 & e_2^- + e_1^+ + 1 & e_2 + 1 & e_5 + 1 & e_1^- + e_5^+ + 1 & 0 \end{pmatrix}$$

If there exists a thrackle drawing of K_5 on M_2, then as $\dim \mathrm{H}_1(M_2) = 4$, one has $\operatorname{rank} \Omega_{M_2} = 4$. We consider rotation systems for which $\operatorname{rank} \Omega \leq 4$; that is, since Ω has even rank, $\operatorname{rank} \Omega \neq 6$, or equivalently, $\mathrm{Pf}(\Omega) = 0$, where

$$\begin{aligned}\mathrm{Pf}(\Omega) =& \Omega_{1,2}(\Omega_{3,6}\Omega_{4,5} - \Omega_{3,5}\Omega_{4,6} + \Omega_{3,4}\Omega_{5,6}) - \Omega_{1,3}(\Omega_{2,4}\Omega_{5,6} - \Omega_{2,5}\Omega_{4,6} + \Omega_{2,6}\Omega_{4,5}) \\ &+ \Omega_{1,4}(\Omega_{2,6}\Omega_{3,5} - \Omega_{2,5}\Omega_{3,6} + \Omega_{2,3}\Omega_{5,6}) - \Omega_{1,5}(\Omega_{2,6}\Omega_{3,4} - \Omega_{2,4}\Omega_{3,6} + \Omega_{2,3}\Omega_{4,6}) \\ &+ \Omega_{1,6}(\Omega_{2,5}\Omega_{3,4} - \Omega_{2,4}\Omega_{3,5} + \Omega_{2,3}\Omega_{4,5}) \mod 2.\end{aligned}$$

For thrackle drawings, Part (b) of Lemma 3.2 implies that the 3-cycles occupy distinct homology classes; in particular, the kernel of Ω cannot contain the difference of any pair of distinct 3-cycles. We now consider the 50 rotation systems of Table 3, and for each system we check whether $\mathrm{Pf}(\Omega) = 0$ and we check whether the kernel of Ω contains any of the (10 choose 2) differences of distinct 3-cycles kernel. This is a little more work than we had to do for $K_{3,3}$, but the procedure is entirely mechanical. We find:

LEMMA 3.4. *Of the 50 inequivalent rotation systems of K_5 listed in Table 3, only two have* $\operatorname{rank} \Omega_{\mathcal{T}} \leq 4$ *and have the property that distinct 3-cycles belong to distinct homology classes. They are:*

$$(2,3,4,5),\ (3,5,4,1),\ (4,1,5,2),\ (2,1,5,3),\ (3,1,2,4)$$
$$(2,3,4,5),\ (3,5,4,1),\ (5,1,4,2),\ (2,1,5,3),\ (3,1,2,4).$$

and Ω is then respectively:

$$\begin{pmatrix} 0&1&0&0&0&1\\ 1&0&1&0&1&1\\ 0&1&0&0&1&0\\ 0&0&0&0&1&1\\ 0&1&1&1&0&1\\ 1&1&0&1&1&0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0&0&0&1&0&1\\ 0&0&0&0&1&1\\ 0&0&0&1&1&0\\ 1&0&1&0&1&1\\ 0&1&1&1&0&1\\ 1&1&0&1&1&0 \end{pmatrix}.$$

In fact, in both of the above cases, one has $\operatorname{rank} \Omega = 4$. Moreover, in both cases, the 15 4-cycles are all distinct in homology, no 3-cycle is homologous to a 5-cycle, and the 12 5-cycles occupy 6 homology classes (one of which is trivial), each 5-cycle being homologous to its complement in K_5. In the first case, the kernel of Ω consists of the identity element, K_5 itself, and the two 5-cycles whose coordinates are: $(0,1,0,0,1,1), (1,0,1,1,0,0)$. This is the cycle 13254 and its dual. In the second case, the kernel of Ω consists of the identity element, K_5 itself, and the two 5-cycles whose coordinates are: $(0,0,0,1,1,1), (1,1,1,0,0,0)$. This is the cycle 12534 and its dual.

4. Completion of Proof for K_5

Suppose that we have a thrackle drawing $\mathcal{T} : K_5 \to M_2$. Lemma 3.4 shows that there are only two choices for the rotation systems and moreover in each case $\operatorname{rank} \Omega_{\mathcal{T}} = 4$ and hence the induced map in $\mathbb{Z}_2$-homology $\mathcal{T}_* : \mathrm{H}_1(K_5) \to \mathrm{H}_1(M_2)$ is surjective. Thus one can choose a basis $\hat{b}_1, \dots, \hat{b}_6$ of $\mathrm{H}_1(K_5)$ such that $\hat{b}_5, \hat{b}_6$ spans the kernel of $\mathcal{T}_*$ and that the elements $b_i = \mathcal{T}_* \hat{b}_i$, for $i = 1, \dots, 4$, spans $\mathrm{H}_1(M_2)$ and relative to this basis the intersection form Ω_{M_2} is represented by the following

First Case	Second Case
$\hat{b}_1 = c_2,$	$\hat{b}_1 = c_2$
$\hat{b}_2 = c_2 + c_3$	$\hat{b}_2 = c_5$
$\hat{b}_3 = c_2 + c_3 + c_5$	$\hat{b}_3 = c_3 + c_4$
$\hat{b}_4 = c_1 + c_6$	$\hat{b}_4 = c_1 + c_3 + c_5 + c_6$
$\hat{b}_5 = c_2 + c_5 + c_6$	$\hat{b}_5 = c_4 + c_5 + c_6$
$\hat{b}_6 = c_1 + c_3 + c_4$	$\hat{b}_6 = c_1 + c_2 + c_3$

TABLE 5. Bases for $\mathrm{H}_1(K_5)$

First Case	Second Case
$p = 13254 = c_2 + c_5 + c_6$	$p = 12534 = c_4 + c_5 + c_6$
$t_1 = 134 = c_1 + c_2 + c_5$	$t_1 = 124 = c_1 + c_3 + c_5$
$t_2 = 123 = c_2$	$t_2 = 125 = c_6$
$t_3 = 235 = c_1 + c_4 + c_6$	$t_3 = 235 = c_1 + c_4 + c_6$
$t_4 = 245 = c_1 + c_3 + c_6$	$t_4 = 345 = c_4$
$t_5 = 145 = c_5$	$t_5 = 134 = c_1 + c_2 + c_5$

TABLE 6. New Bases for $\mathrm{H}_1(K_5)$

matrix:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

For the 2 cases of Lemma 3.4, Table 5 shows choices for the basis $\hat{b}_1, \dots, \hat{b}_6$, relative to the basis $c_1, \dots, c_6$ used in the previous section.

At this point, it is convenient to change our notation. We replace the basis $c_1, \dots, c_6$ of $H_1(G)$ by a basis $p, t_1, t_2, t_3, t_4, t_5$ depending on the two cases; our choice is given in Table 6. In each case, p is a 5-cycle ("p" for pentagon) which is null-homologous in M_2, and the t_i are 3-cycles ("t" for triangle). It is also useful to relabel the vertices of K_5: we replace the labels $1,2,3,4,5$ by $1,3,2,5,4$ respectively, in the first case, and by $1,2,4,5,3$ respectively in the second case. Thus in both cases, p is now the 5-cycle 12345 and the t_i are the 3-cycles: $t_i = i(i+1)(i-1)$, $i = 1, \dots, 5$, where the numbers are taken modulo 5. Note that we are regarding $p, t_1, \dots, t_5$ as elements of the $\mathbb{Z}_2$-homology $H_1(G)$ and so their orientation is irrelevant.

By abuse of language, we will also use the symbols $p, t_1, \dots, t_5$ to denote the images of the respective cycles in $\mathrm{H}_1(M_2)$. In passing from $\mathrm{H}_1(K_5)$ to $\mathrm{H}_1(M_2)$, the $\hat{b}_i$ go to b_i, for $i = 1, \dots, 4$, and $\hat{b}_5, \hat{b}_6$ go to 0, and the equations of Tables 5 and 6 enable us to express $p, t_1, \dots, t_5$ in terms of the basis $b_1, \dots, b_4$; the resulting equations are shown in Table 7.

Consider the graph G obtained from K_5 by adding a vertex at each self-intersection point of the image of the 5-cycle $p' = 13524$ under the map $\mathcal{T}$. (Note that $p' = t_1 + \cdots + t_5$; p' is dual to $p = 12345$, and like p, it is null-homologous in $H_1(M_2)$.) The graph G has 10 vertices and 20 edges, and thus $\mathrm{H}_1(G)$ has dimension 11. The cycles $p, t_1, \dots, t_5$ determine elements of $\mathrm{H}_1(G)$, and we extend this set of

First Case	Second Case
$p = 0$	$p = 0$
$t_1 = b_4$	$t_1 = b_1 + b_2$
$t_2 = b_1$	$t_2 = b_1 + b_2 + b_4$
$t_3 = b_3$	$t_3 = b_2 + b_3 + b_4$
$t_4 = b_1 + b_2 + b_4$	$t_4 = b_1 + b_4$
$t_5 = b_2 + b_3$	$t_5 = b_1 + b_2 + b_3 + b_4$

TABLE 7. Equations for Basis Elements

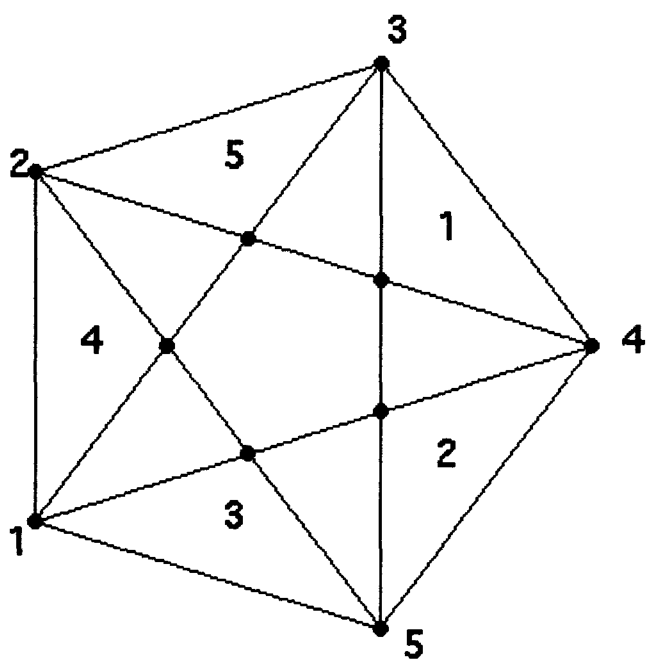

FIGURE 2. One possibility for G

elements to a basis for $H_1(G)$ by introducing elements $\hat{f}_1, \ldots, \hat{f}_5$ as follows. Let x_i denote the intersection of edge $(i+1)(i+3)$ with edge $(i-1)(i-3)$. Then $\hat{f}_i$ is defined to be the 3-cycle $(i+2)(i-2)x_i$. Figure 2 may assist the reader in visualizing the situation; in this figure, $\hat{f}_i$ is the 3-cycle which bounds the face denoted i. (However, we stress that we are not assuming that on the edges of G, the order of intersections and their orientations are those depicted in Fig. 2.) Let f_i denote the image of $\hat{f}_i$ in $H_1(M_2)$.

We use the basis $b_1, \ldots, b_4$ to identify $H_1(M_2)$ with $\mathbb{Z}_2^4$, and for vectors $x, y \in H_1(M_2)$, we denote by (x, y) the intersection number xFy^t, where F is the matrix of equation (4.1) and y^t denotes the transpose of y.

We now consider some equations that follow from the fact that our drawing is a thrackle. Consider the path $x_1 3425$ in G; this is just the 4-cycle $3425 = p + t_1 + t_3 + t_4$. It follows that since edge 25 crosses edge 34 precisely once, the paths $x_1 34$ and $25x_1$

First Case	Second Case
$f_{1,1} = 1 + f_{1,2} + f_{1,4}$	$f_{1,1} = f_{1,2}$
$f_{1,3} = 1$	$f_{1,4} = 1$
$f_{2,2} = 0$	$f_{2,1} = 1 + f_{2,4}$
$f_{2,3} = 1 + f_{2,4}$	$f_{2,2} = 1 + f_{2,3} + f_{2,4}$
$f_{3,1} = 1 + f_{3,3}$	$f_{3,1} = 1$
$f_{3,4} = 1$	$f_{3,3} = 1 + f_{3,4}$
$f_{4,1} = 1$	$f_{4,2} = 1$
$f_{4,2} = 1 + f_{4,3}$	$f_{4,3} = 1$
$f_{5,1} = f_{5,4}$	$f_{5,1} = 1 + f_{5,3}$
$f_{5,2} = 1 + f_{5,4}$	$f_{5,2} = 1 + f_{5,4}$

TABLE 8. Equations in $\mathrm{H}_1(M_2)$

have intersection number 1, and hence the paths $x_1 34$ and 3425 have intersection number 1; that is, $(f_1, p + t_1 + t_3 + t_4) = 1$. Similarly, for all $i = 1, \dots, 5$, one has:

$$(f_i, p + t_i + t_{i+2} + t_{i-2}) = 1. \tag{4.2}$$

The same argument can be made concerning the intersection of the paths $x_1 45$ and $2354 (= t_3 + t_4)$. As $x_1 45 = f_1 + t_4$, this gives $(f_1 + t_4, t_3 + t_4) = 1$, and more generally, for all $i = 1, \dots, 5$, one has:

$$(f_i + t_{i-2}, t_{i+2} + t_{i-2}) = 1,$$

or equivalently,

$$(f_i, t_{i+2} + t_{i-2}) = 1 + (t_{i-2}, t_{i+2}). \tag{4.3}$$

Denote the components of f_i relative to the basis $b_1, \dots, b_4$ by $f_{i,j}$, for $j = 1, \dots, 4$. Using the values of t_i from Table 6, with the basis vectors $b_1 = (1,0,0,0), b_2 = (0,1,0,0)$, etc, and using the definition of the bracket $(\, , \,)$, the 10 equations (4.2) and (4.3) are linear in the 20 variables $f_{i,j}$, thus allowing us to eliminate half of them. The resulting equations are shown in Table 8. On the face of it, there are 2^{10} remaining possibilities for the system of variables $f_{i,j}$.

We have now gone as far as we can without considering the order of intersection of the edges. We will restrict our attention initially to the order of the self-intersections of the null-homologous cycle $p' = 13524$. But first, we need a general result on null-homologous curves. Recall the definition of the *Gauss word* of a curve. Suppose that γ is a closed curve on some surface M_g and that γ is *normal*; i.e., γ has some finite number n of distinct self-intersections, each of which is a transverse crossing. Label the crossing points $a, b, c, \dots$, and then, as one travels along γ, starting at some arbitrary point, write down the ordered list of crossing points that one encounters as one makes one complete tour of γ. The resulting word has $2n$ letters (each letter occurring twice) and is called the Gauss word of γ. *Gauss' condition* is that for each letter x say, there is an even number of letters in the Gauss word between the two occurrences of x. As is well known, Gauss' condition is a necessary but not sufficient condition for a given word to be the Gauss word of a planar curve; see [**6, 1, 2**]. It would not be surprising if the following had been previously observed, though we are not aware of it in the literature.

THEOREM 4.1. *Suppose that γ is a closed normal curve on an oriented surface M_g. If γ is null-homologous in $\mathbb{Z}_2$-homology, then γ satisfies Gauss' condition.*

Moreover, the converse holds when γ is cellular, in the sense that its complement is homeomorphic to a disjoint union of discs.

PROOF. Note that γ is null-homologous if and only if M_g can be coloured with two colours (black and white, say) such that the colours change across γ, and only across γ; in other words, γ is the boundary of the white (and black) region. If γ is null-homologous, and M_g is a correspondingly coloured, then Gauss' condition holds as one can see by starting at an arbitrary crossing x, and looking at the colour on the left of γ as one travels along γ from x to x.

To see the converse, suppose that γ is cellular;, the complement of γ is thus a disjoint union of discs, which we call *faces.* We make the following construction. For each face f, choose a point p_f in the interior of f and draw lines from p_f to each of the vertices of f. Clearly we may choose these lines so that they are mutually disjoint, other than at the point p_f. We have thus subdivided f into a number of triangular regions, each of which has p_f as a vertex and has a segment of γ as base. (We allow degenerate triangles to deal with the case of a face bound by a simple loop; here the face is composed of a single triangle, two of whose edges coincide). In particular, at each non-vertex point of γ there is a well defined "triangle to the left of γ", and an equally well defined "triangle to the right of γ". Now consider the following colouring of the triangles. Start at some non-vertex point x on γ and colour the triangle to the left of γ at x black, and colour the triangle to the right white. Now travel along γ (in a fixed direction) and colour the triangles to the left and right of γ according to the following rule: the colours of both left and right triangles are changed following each self-intersection of γ. Stop the process once all triangles have been coloured. In any particular face the colours of the triangles agree if and only if the colours of adjacent triangles agree at the vertices. But if x is an arbitrary vertex, and if T_1 and T_2 are two adjacent triangles which both have x as a common vertex, then T_1 and T_2 have the same colour if and only if one encounters an even number of self-intersections of γ as one travels along γ from x to x. Consequently, if Gauss' condition holds, the faces are monochromatic and so γ is null-homologous. □

Now consider the crossings on the thrackle of an arbitrary 5-cycle c. The crossings of c are conveniently recorded by using its *intersection table.* Choose as orientation for c and, starting at some arbitrary vertex, label the consecutive edges of c with the numbers $1, \ldots, 5$. The i^{th}-row of the intersection table consists of the numbers of the edges that the i^{th}-edge crosses as it is traversed in the positive direction. Using the above theorem, the following lemma can be easily verified by hand:

LEMMA 4.2. *If c is a null-homologous thrackle of the 5-cycle, then the intersection table of c is one of the following two possibilities:*

3 4	4 3
4 5	5 4
5 1	1 5
1 2	2 1
2 3	3 2.

Note that the above lemma implies that the order of the crossings of our null-homologous 5-cycle p are either the same as those shown in Fig. 2, or they are the

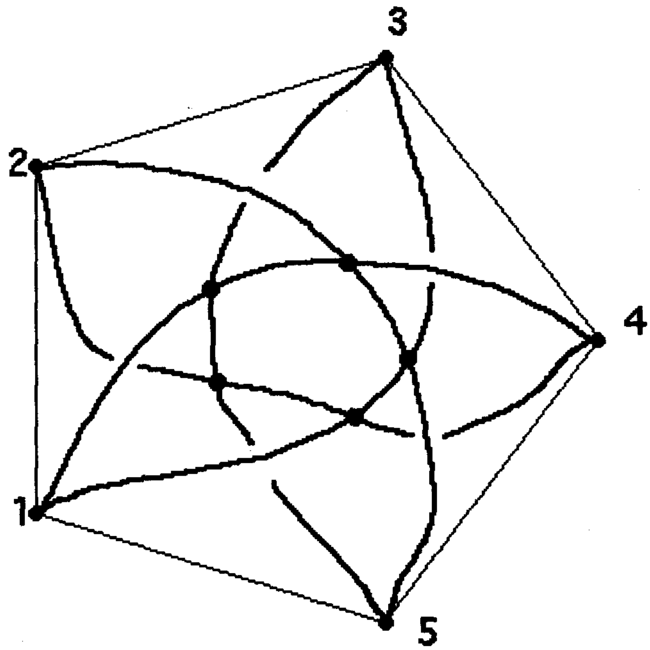

FIGURE 3. Another possibility for G

opposite order on each edge, as in the example shown in Fig. 3. (Tables with such cyclic symmetry are called *mushquash*; see [**3, 4**].)

We first suppose that the orders of the crossings are as shown in Fig. 2. Let us consider (f_1, f_2). The intersection number between f_1 and f_2 is determined by the rotation system at their common vertex 4, and this rotation system is itself encoded in the intersection of the cycles $234(=t_3)$ and $145(=t_5)$. Explicitly, $(f_1, f_2) = 1 + (t_3, t_5)$. More generally,

$$(f_i, f_{i+1}) = 1 + (t_{i+2}, t_{i-1}), \tag{4.4}$$

for all $i = 1, \ldots, 5$. Notice also that $f_1+f_2+t_4$ and $f_2+f_3+t_5$ have zero intersection. Similarly,

$$(f_i + f_{i+1} + t_{i-2}, f_{i+1} + f_{i+2} + t_{i-1}) = 0, \tag{4.5}$$

for all $i = 1, \ldots, 5$. Combining Equations (4.4) and (4.5) with the equations listed in Table 8, one obtains a system which has only 2 solutions in the first case (the left side of Table 8), and no solutions in the second case (the right side of Table 8). The two remaining possibilities in the first case are:

$$\begin{aligned}
f_1 &= b_1 + b_2 + b_3 + b_4 \\
f_2 &= b_3 \text{ or } b_4 \\
f_3 &= b_2 + b_3 + b_4 \\
f_4 &= b_1 + b_2 + b_4 \\
f_5 &= b_2 + b_3.
\end{aligned}$$

To eliminate these last two possibilities, consider the intersection of edge 45 with edge 13. This intersection must occur in the boundary of precisely one of either $t_1 + f_4$, $t_2 + f_4 + f_5$, or $t_3 + f_5$. That is, precisely one of the numbers $(f_2, t_1 + f_4) + (t_4, t_1) + 1, (f_2, t_2 + f_4 + f_5)$ and $(f_2, t_3 + f_5) + (t_3, t_5) + 1$ must equal 1. However, one finds that in both of the above possibilities, all three numbers are 1.

It remains to treat the case where the orders of the crossings are as shown in Fig. 3. Once again, consider (f_1, f_2). This time, one has $(f_1 + t_4, f_2 + t_4) = (t_3, t_5)$, or equivalently, $(f_1, f_2) = (f_1 + f_2, t_4) + (t_3, t_5)$, and more generally,

$$(f_i, f_{i+1}) = (f_i + f_{i+1}, t_{i-2}) + (t_{i+2}, t_{i-1}) \tag{4.6}$$

for all $i = 1, \ldots, 5$. Notice also that $p+t_2+t_5+t_1+f_3+f_4$ and $p+t_4+t_2+t_3+f_5+f_1$ have intersection number 1, and more generally,

$$(p + t_{i+1} + t_{i-1} + t_i + f_{i+2} + f_{i-2}, p + t_{i-2} + t_{i+1} + t_{i+2} + f_{i-1} + f_i) = 1, \tag{4.7}$$

for all $i = 1, \ldots, 5$. Combining Equations (4.6) and (4.7) with the equations listed in Table 8, one obtains a system which has no solution in either of the cases of Table 8. This completes our study of K_5.

5. Completion of Proof for $K_{3,3}$

Our treatment of $K_{3,3}$ is similar to that employed for K_5 in the previous section, though the computations are more demanding. For $K_{3,3}$, Lemma 3.3 gives a single case to consider, whereas for K_5, Lemma 3.4 gave two cases. However, while we saw in Lemma 4.2 that there were only two possible intersection tables for a null-homologous thrackle of the 5-cycle, there are considerably more possibilities for the 6-cycle. Indeed, if *abc* is the i^{th}-row of the intersection table of a null-homologous thrackle of the 6-cycle, then the new table obtained by leaving the other rows unchanged and replacing the i^{th}-row by *cba* is again the intersection table of a null-homologous thrackle of the 6-cycle (since Gauss' condition is unchanged). The same holds if one interchanges the first and last letter in more than one row. Thus starting from a given table, one can construct 2^6 tables, which we regard as being *equivalent* to the given table. Note that if the crossings are numbered $1, \ldots 6$, then every table is equivalent to one in which, in every row, the first number is less than the last number. These tables are easily determined, and this gives:

LEMMA 5.1. *If c is a null-homologous thrackle of the 6-cycle, then the intersection table of c is one of the $2^8 = 256$ tables equivalent to one of following 4 possibilities:*

3 5 4	3 5 4	4 3 5	4 3 5
4 6 5	5 4 6	5 4 6	4 6 5
5 1 6	5 1 6	1 5 6	1 5 6
1 2 6	1 6 2	1 6 2	1 2 6
1 3 2	1 3 2	2 1 3	2 1 3
2 4 3	3 2 4	3 2 4	2 4 3.

Before using Lemma 5.1, let us make some general comments. For the unique rotation system of Lemma 3.3, one has $\operatorname{rank} \Omega_{\mathcal{T}} = 2$ and hence the induced map in $\mathbb{Z}_2$-homology $\mathcal{T}_* : \mathrm{H}_1(K_{3,3}) \to \mathrm{H}_1(\mathbb{T})$ is surjective. Consider the basis $c_1, \ldots, c_4$ for

$\mathrm{H}_1(K_{3,3})$ used in Section 3. For the basis

$$\begin{aligned}
\hat{b}_1 &= c_1 + c_2 && (= 123456)\\
\hat{b}_2 &= c_2 + c_3 + c_4 && (= 145236)\\
\hat{b}_3 &= c_2 + c_4 && (= 143256)\\
\hat{b}_4 &= c_1 + c_3 && (= 123654)
\end{aligned}$$

of $\mathrm{H}_1(K_{3,3})$, one sees that $\hat{b}_3, \hat{b}_4$ spans the kernel of $\mathcal{T}_*$ and that the elements $b_i = \mathcal{T}_* \hat{b}_i$, for $i = 1, 2$, spans $\mathrm{H}_1(M_2)$. By abuse of language, we also use the symbols $c_1, \dots, c_4$ for the image of these cycles in $\mathrm{H}_1(M_2)$; thus

$$\begin{aligned}
c_1 &= c_3 = b_2,\\
c_2 &= c_4 = b_1 + b_2.
\end{aligned}$$

Relative to this basis b_1, b_2, the intersection form Ω_{M_2} is represented by the following matrix:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

As in Section 4, we denote by (x, y) the intersection number xFy^t, where F is the matrix of equation (5.1) and y^t denotes the transpose of y.

As we saw in Section 3, $K_{3,3}$ has nine 4-cycles; for $i \in \{1,3,5\}, j \in \{2,4,6\}$, let r_{ij} denote the unique 4-cycle ("r" for rectangle) which meets neither vertex i nor j. One has:

$$\left\{\begin{aligned}
r_{12} &= 3456 = c_3 && = b_2\\
r_{14} &= 3256 = c_3 + c_4 && = b_1\\
r_{16} &= 3254 = c_4 && = b_1 + b_2\\
r_{32} &= 1456 = c_2 && = b_1 + b_2\\
r_{34} &= 1256 = c_1 + c_2 + c_4 && = b_2\\
r_{36} &= 1254 = c_1 + c_4 && = b_1\\
r_{52} &= 1436 = c_2 + c_3 && = b_1\\
r_{54} &= 1236 = c_1 + c_2 + c_3 && = b_1 + b_2\\
r_{56} &= 1234 = c_1 && = b_2.
\end{aligned}\right. \tag{5.2}$$

We now consider the three null-homologous 6-cycles $\gamma_1 = 143256, \gamma_2 = 145623, \gamma_3 = 436125$. We label the edges of each of these curves $1, \dots, 6$, in the obvious way; for example, on γ_1, edge 1 is 14, edge 2 is 43, etc. If i is an integer, then edge i of γ_k is defined to be edge i' of γ_k, where i' the unique element of $\{1, \dots, 6\}$ which is congruent to i modulo 6.

For convenience, we introduce a different notation for the 4-cycles, which is adapted to the cycles γ_k. For $i = 1, \dots, 6$, r_i^k is the unique 4-cycle containing edges $i-1, i, i+1$ of γ_k, and for $i = 1, 2, 3$, r_{6+i}^k is the unique 4-cycle containing edges i and $i+3$ of γ_k. See Table 9.

For $i = 1, \dots, 6$, let x_i^k denote the crossing of edges $i+1$ and $i-1$ of γ_k, and let f_i^k denote the cycle comprised of edge i of γ_k and the segments of edges $i+1$ and $i-1$ of γ_k joining x_i to the vertices of edge i. For $i = 1, 2, 3$, let y_i^k denote the crossing of edges i and $i+3$ of γ_k, let η_i^k denote the edge of $K_{3,3}$ joining the initial vertices of edges i and $i+3$ of γ_k, and let f_{6+i}^k denote the cycle comprised of η_i^k and the segments of edges i and $i+3$ of γ_k joining x_i to the vertices η_i^k. (The η_i^k are edges of both cycles γ_j, for $j \neq k$.)

$r_1^1 = r_{52}$	$r_1^2 = r_{36}$	$r_1^3 = r_{12}$
$r_2^1 = r_{56}$	$r_2^2 = r_{32}$	$r_2^3 = r_{52}$
$r_3^1 = r_{16}$	$r_3^2 = r_{12}$	$r_3^3 = r_{54}$
$r_4^1 = r_{14}$	$r_4^2 = r_{14}$	$r_4^3 = r_{34}$
$r_5^1 = r_{34}$	$r_5^2 = r_{54}$	$r_5^3 = r_{36}$
$r_6^1 = r_{32}$	$r_6^2 = r_{56}$	$r_6^3 = r_{16}$
$r_7^1 = r_{36}$	$r_7^2 = r_{52}$	$r_7^3 = r_{56}$
$r_8^1 = r_{12}$	$r_8^2 = r_{16}$	$r_8^3 = r_{14}$
$r_9^1 = r_{54}$	$r_9^2 = r_{34}$	$r_9^3 = r_{32}$

TABLE 9. 4-cycles

$f_2^2 = f_6^1 + r_6^1$	$f_4^2 = f_4^1 + r_4^1$	$f_6^2 = f_2^1 + r_2^1$
$f_2^3 = f_1^1 + r_1^1$	$f_4^3 = f_5^1 + r_5^1$	$f_6^3 = f_3^1 + r_3^1$
$f_1^3 = f_3^2 + r_3^2$	$f_3^3 = f_5^2 + r_5^2$	$f_5^3 = f_1^2 + r_1^2$

TABLE 10. Relations

$f_{12}^1 = 1$	$f_{12}^2 = 1$	$f_{11}^3 = 1$
$f_{21}^1 = 1$	$f_{21}^2 = 1 + f_{22}^2$	$f_{22}^3 = 1$
$f_{31}^1 = 1 + f_{32}^1$	$f_{31}^2 = 1$	$f_{31}^3 = 1 + f_{32}^3$
$f_{42}^1 = 1$	$f_{42}^2 = 1$	$f_{41}^3 = 1$
$f_{51}^1 = 1$	$f_{51}^2 = 1 + f_{52}^2$	$f_{52}^3 = 1$
$f_{61}^1 = 1 + f_{62}^1$	$f_{61}^2 = 1$	$f_{61}^3 = 1 + f_{62}^3$
$f_{72}^1 = 1$	$f_{72}^2 = 1$	$f_{71}^3 = 1$
$f_{81}^1 = 1$	$f_{81}^2 = 1 + f_{82}^2$	$f_{82}^3 = 1$
$f_{91}^1 = 1 + f_{92}^1$	$f_{91}^2 = 1$	$f_{91}^3 = 1 + f_{92}^3$

TABLE 11. Equations

There are 27 f_i^k, but there is a certain redundancy in this notation; Table 10 gives 9 relations, thus reducing the system to 18 independent cycles:

$$f_1^1, \dots, f_9^1, f_3^2, f_5^2, f_1^2, f_7^2, f_8^2, f_9^2, f_7^3, f_8^3, f_9^3. \tag{5.3}$$

Though we won't use this fact, we remark that, along with the cycles $c_1, \dots, c_4$, these 18 cycles form a basis for the first homology group of the graph obtained from $K_{3,3}$ by adding a vertex at each crossing.

The 4-cycles r_i^k have been defined so that for all $k \in \{1,2,3\}, i \in \{1,\dots,9\}$, the edges of the cycle f_i^k form part of the edges of r_i^k. For example, $r_1^1 = 1436$, while $f_1^1 = 14x_1^1$, where x_1^1 is the intersection of edges 34 and 16. It follows that f_1^1 intersects $f_1^1 + r_1^1 = 36x_1^1$ exactly once; thus $(f_1^1, r_1^1 + f_1^1) = 1$ and hence $(f_1^1, r_1^1) = 1$. More generally,

$$(f_i^k, r_i^k) = 1, \tag{5.4}$$

for all $k \in \{1,2,3\}, i = 1,\dots,9$. Each f_i^k has two components with respect to the basis b_1, b_2; let us denote this $\mathbb{Z}_2$-vector (f_{i1}^k, f_{i2}^k). Using the identities of (5.2) and Table 9, Equation (5.4) can be expressed in terms of the components f_{i1}^k, f_{i2}^k; the results are shown in Table 11. Notice that the 18 cycles listed in (5.3) have altogether $18 \times 2 = 36$ components of which 18 may be regarded are independent variables, using Table 11.

We now consider some relations that depend on the order of the self-intersections of cycles $\gamma_1, \gamma_2, \gamma_3$, but not on the orientation of these crossings.

NOTATION 5.2. If i is an integer, and j is a natural number, we denote by $[i]_j$ the unique element of $\{1, \ldots, j\}$ which is congruent to i modulo j.

We introduce a variable $\epsilon_i^k(j,l) \in \{0,1\}$, which we set equal to 1 when on γ_k, edge i encounters edge j before edge l. In terms of the intersection table of γ_k, one has:

$$\epsilon_i^k(j,l) = \begin{cases} 1, & \text{if } j \text{ is before } l \text{ in row } i \\ 0, & \text{otherwise,} \end{cases}$$

for all $i,j,l \in \{1,\ldots,6\}$. For convenience, we extend $\epsilon_i^k(j,l)$ to integer values of i,j,l by setting $\epsilon_i^k(j,l) = \epsilon_{[i]_6}^k([j]_6,[l]_6)$.

For $i = 1,\ldots,6$, consider the intersection of f_i^k with $f_{[i+3]_6}^k$. A careful examination shows that:

(5.5)

$$\begin{aligned}(f_i^k, f_{[i+3]_6}^k) = \ & \epsilon_{i+1}^k(i+3,i-1) + \epsilon_{i-1}^k(i+1,i+3) + \epsilon_{i+2}^k(i+4,i) + \epsilon_{i+4}^k(i,i+2) \\ & + \epsilon_{i+1}^k(i+4,i-1).\epsilon_{i+4}^k(i+1,i+2) \\ & + \epsilon_{i-1}^k(i+1,i+2).\epsilon_{i+2}^k(i+4,i-1),\end{aligned}$$

for all $k \in \{1,2,3\}, i = 1,\ldots,6$. Similarly, for $i = 1,\ldots,6$, one can consider the intersection of f_i^k and $f_{[i+2]_6}^k$. If on edge $i+1$ of γ_k, the crossing of edge $i-1$ precedes the crossing of edge $i+3$, then the number (f_i^k, f_{i+2}^k) does not depend upon the orientation of the crossings. Indeed, in this case, one has:

$$(f_i^k, f_{[i+2]_6}^k) = 1 + \epsilon_{i-1}^k(i+1,i+2) + \epsilon_{i+3}^k(i,i+1) + \epsilon_{i+3}^k(i-1,i).$$

We can express this as an equation which is valid in all cases, as follows:

(5.6)

$$\epsilon_{i+1}^k(i-1,i+3).\left(\begin{array}{ll} (f_i^k, & f_{[i+2]_6}^k) + 1 + \epsilon_{i-1}^k(i+1,i+2) \\ + & \epsilon_{i+3}^k(i,i+1) + \epsilon_{i+3}^k(i-1,i).\epsilon_{i-1}^k(i+2,i+3) \end{array}\right) = 0$$

for all $k \in \{1,2,3\}, i = 1,\ldots,6$. For $i = 1,2,3$, one can consider the intersection of f_{6+i}^k with the cycle $r_{6+[i-1]_3}^k + f_{6+[i-1]_3}^k$, which shares the edge η_i^k in common with f_{6+i}^k. One finds:

(5.7)

$$\begin{aligned}(f_{6+i}^k, r_{6+[i-1]_3}^k + f_{6+[i-1]_3}^k) = \ & 1 + (r_{i+4}^k, r_{i+1}^k) + \epsilon_i^k(i+2,i+3).\epsilon_{i+2}^k(i-1,i) \\ & + \epsilon_{i+3}^k(i-1,i).\epsilon_{i-1}^k(i+2,i+3),\end{aligned}$$

for all $k \in \{1,2,3\}, i = 1,2,3$.

Notice that if the orders of intersection are given, we may regard Equations (5.5), (5.6), (5.7) as a system of 45 quadratic equations over $\mathbb{Z}_2$ in the 18 independent variables f_{il}^k.

By Lemma 5.1, there are 256 possibilities for the intersection table of each of the null-homologous curves γ_k. However, γ_k shares 3 common edges with each of the curves γ_l, $l \neq k$. This gives compatibility conditions between the three intersection tables. We find that for each intersection table for γ_1, there are 128 pairs of intersection tables for γ_2, γ_3 which are compatible with each other and with the given table for γ_1. Thus, in all, there are $256 \times 128 = 2^{15}$ triples of mutually compatible intersection tables for the null-homologous curves $\gamma_1, \gamma_2, \gamma_3$.

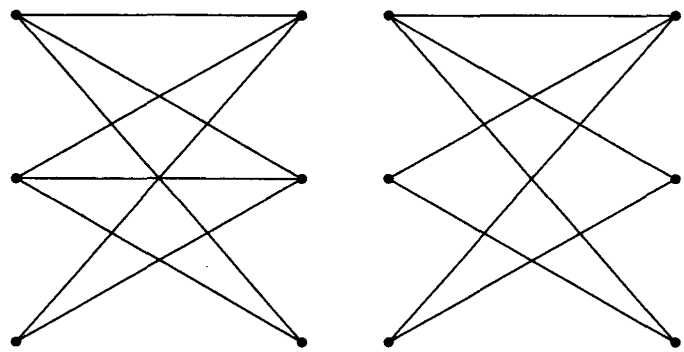

FIGURE 4

(The interested reader may download a list of these triples from the first author's web page: http://Grant-Cairns.maths.latrobe.edu.au/. We remark that a choice for the 3 tables does not necessarily uniquely determine the order of the 4 intersections of $K_{3,3}$ on each edge.)

It remains to run through this list and in each case, verify that the Equations (5.5), (5.6), (5.7) have no common solution over $\mathbb{Z}_2$. This took us approximately 12 hours, running Mathematica on a Mac G4. As the system has no solution, we conclude that $K_{3,3}$ cannot be thrackled on $\mathbb{T}$.

6. Proof of Corollary 1.2

The thrackle conjecture was established in [**5**] for graphs apart from K_5 with ≤ 5 vertices. So Theorem 1.1 completes the study of graphs with ≤ 5 vertices. Apart from $K_{3,3}$, there are only three bipartite graphs that have 6 vertices with no terminal vertex, and that have more edges than vertices; they are $K_{2,4}$, which has 8 edges, and the two graphs shown in Fig. 4, which have 8 and 7 edges respectively. Since all three graphs contain 4-cycles, none can be thrackled on the plane, by Lemma 3.2(a). This completes the proof of the corollary.

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